RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

ONTO INNER DERIVATIONS IN DIVISION RINGS

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1. Introduction. Kaplansky [3] proposed the following problem: Does there exist a division ring $\Delta$ each element of which is a sum of additive commutators $ab - ba$? In [1] Harris gave a strongly affirmative solution to this problem by constructing division rings $\Delta$ in which each element $c = ab - ba$ for some $a, b \in \Delta$. Recently Meisters [4] has studied rings $R(\Delta)$ in which for any triple of elements $a, b, c \in R$ with $a \neq b$ there exist solutions of the equation $ax - xb = c$. He has shown that (1) $R$ is a division ring in which every noncentral element induces an onto inner derivation and (2) if $R$ is separable algebraic over its center, then $R$ is commutative. Actually one can prove the more general result that in a division ring $R$ of the preceding type all algebraic elements (over the center) are central. (Hence if $R$ is noncommutative, each noncentral element $t \in R$ is transcendental over the center of $R$ and induces an onto inner derivation.)

In view of the above work it seems natural to investigate the question of existence of division rings possessing onto inner derivations. We give a partial answer to this question which implies (in some heuristic sense) that Harris' examples (at least for char. $p > 0$) are normative rather than pathological. More precisely we sketch a proof of the following theorem: For each division ring $\Delta$ of char. $p > 0$ one can construct an extension division ring $E$ with the property that there exists an element $t \in E$ (lying in the centralizer of $\Delta$) whose associated inner derivation $D_t$ is an onto map: $D_t(E) = E$.

2. Preliminaries. We shall make consistent use of the following facts: (1) Any noncommutative ring $R$ with an identity having the common right multiple property has a right quotient ring $Q(R)$, i.e., every element of $Q(R)$ has the form $ab^{-1}$, $a, b \in R$, $b$ regular, and all regular elements of $R$ are invertible in $Q(R)$. (2) If $\Delta$ is a division ring and $D$ a derivation of $\Delta$ into itself, then $\Delta[x; D]$, the ring of differential polynomials over $\Delta$ in the indeterminate $x$, has the com-
mon right multiple property; thus by (1), \( \Delta[x; D] \) has a quotient division ring \( Q(\Delta[x; D]) \), since all nonzero elements in \( \Delta[x; D] \) are regular. (3) If \( R \) is a ring with quotient ring \( Q(R) \) and \( D \) is a derivation of \( R \) into an extension ring \( S \) of \( Q(R) \), then \( D \) can be uniquely extended to a derivation of \( Q(R) \) into \( S \) by defining, for \( ab^{-1} \in Q(R) \),

\[
D(ab^{-1}) = D(a)b^{-1} - (ab^{-1})(D(b)b^{-1}).
\]

A proof of (1) may be found in [2, p. 118]; (2) was established in [5]; and (3) is a fairly straightforward exercise in computation. Finally note that in rings of char. \( p > 0 \) all \( p^n \)th powers (\( n \geq 0 \)) of a derivation are again derivations.

3. The construction. Let \( \Delta_0 \) be the quotient division ring of the polynomial ring \( \Delta[t] \) (\( \Delta \) a division ring of char. \( p > 0 \)) where \( t \) is a commuting indeterminate over \( \Delta \). Set \( x_0 = 1 \) and let \( D_0 \) be the unique extension of ordinary differentiation in \( \Delta[t] \) to \( \Delta_0 \) so that \( D_0 \) is a derivation of \( \Delta_0 \) into itself. Choose an indeterminate \( x_1 \) over \( \Delta_0 \) and form the quotient division ring \( \Delta_1 = Q(\Delta_0[x_1; D_0]) \). Noting that \( D_1(x_1) = x_0 \) and \( D_0(x_0) = 0 \), we see that we have verified the case \( n = 0 \) of the proposition: Given \( \Delta_0 = Q(\Delta[t]) \) there exists a nested sequence of division rings \( \Delta_n \), a set of derivations \( D_n: \Delta_n \rightarrow \Delta_n \), and elements \( x_n \in \Delta_n \) satisfying

1. \( \Delta_{n+1} = Q(\Delta_n[x_{n+1}; D_n]) \),
2. \( D_1(x_{n+1}) = x_n \),
3. \( D_n(t) = x_n \), \( D_n(x_i) = 0 \), \( i = 0, \ldots, n; n \geq 0 \).

To prove this proposition we proceed by induction. Suppose the truth of the proposition for \( n = 0, \ldots, s \). Then we have constructed \( \Delta_n, D_n, x_n \) for \( n = 0, \ldots, s \), satisfying the above conditions. Choose an indeterminate \( x_{s+1} \) over \( \Delta_s \) and let \( \Delta_{s+1} = Q(\Delta_s[x_{s+1}; D_s]) \). We must construct a derivation \( D_{s+1}: \Delta_{s+1} \rightarrow \Delta_{s+1} \) satisfying \( D_{s+1}(t) = x_{s+1} \), \( D_{s+1}(x_i) = 0 \) (\( i = 0, \ldots, s+1 \)), and \( D_1(x_{s+1}) = x_n \). We do this by defining \( D_{s+1} \) on \( \Delta_0 \) and extending it to each successive \( \Delta_i \) (\( i = 1, \ldots, s+1 \)) as follows. Suppose \( D_{s+1} \) has been defined on \( \Delta_i \), \( 0 \leq i < s+1 \); then to define it on \( \Delta_{i+1} \) we need only check that it can be extended to \( \Delta_{i+1}[x_{i+1}; D_i] \). Now if \( \sum a_i x_{i+1}^i, a_i \in \Delta_i \), is a typical element of this ring we set \( D_{s+1}(\sum a_i x_{i+1}^i) = \sum D_{s+1}(a_i)x_{i+1}^i \). Since the map \( D_{s+1}D_i - D_iD_{s+1} \) is zero on \( \Delta_i \), one verifies that \( D_{s+1} \) as defined is a derivation on \( \Delta_{i+1} \). Thus if \( D_{s+1} \) can be constructed on \( \Delta_0 \) we shall be done. Let \( a \in \Delta[t] \). Define

\[
D_{s+1}(a) = \sum_{i=0}^{s+1} D_s^{i+1}(a)/(i+1)! x_{i+1} - i \pmod{p}.
\]
This makes sense since the coefficients of $D_l^{i+1}(a)$ are divisible by $(i+1)!$. Observing that $x_l a = \sum_{i=0}^{i-1} D_l^i(a)/i! \cdot x_{l-1} \pmod{p}$, $l = 0, \ldots, s+1$, one verifies that $D_{s+1}$ is a derivation on $\Delta[t]$ and hence on $\Delta_0$. By what we have said previously it has an extension to $\Delta_{s+1}$ and clearly satisfies all requisite properties.

Next let $E = \bigcup_{n=0}^{\infty} \Delta_n$. Since $D_i(x_n) = x_{n-1}$ we get $D_l^{n+1}(x_n) = 0$ and therefore there exists a least integer $l \geq 0$ for which $D_l^{n}(x_n) = 0$. It is immediate that $D_l^{n}(\Delta_n) = 0$, so $\Delta_n$ is contained in the centralizer of $t^n$. But $D_l^{n}(x_p) = 1$, hence if $a$ is in the centralizer of $t^n$, $x^n a t^n - t^n x^n a = a$. It follows, since $x^n a$ is in $\Delta_n$, that $D_l^{n}(\Delta_n) \supseteq \Delta_n$. But $D_l^{n}(\Delta_n) \supseteq D_l^{n}(\Delta_{n+i}) \supseteq \Delta_n$. As $n$ was arbitrary, $D_l(E) = E$.

**References**


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