In this note we will be concerned with the proof and consequences of the following fact: if \( \phi_0 \) is a differentiable action of a compact Lie group on a compact differentiable manifold \( M \), then any differentiable action of \( G \) on \( M \) sufficiently close to \( \phi_0 \) in the \( C^1 \)-topology is equivalent to \( \phi_0 \).

1. Notation. In what follows differentiable means class \( C^\infty \). If \( M \) and \( V \) are differentiable manifolds, \( \mathfrak{M}(M, V) \) is the space of differentiable maps of \( M \) into \( V \) in the \( C^K \)-topology where \( K \) is a positive integer or \( \infty \) fixed throughout. We denote by \( \text{Diff}(M) \) the group of automorphisms of \( M \) topologized as a subspace of \( \mathfrak{M}(M, M) \). As such it is a topological group. \( \mathfrak{D}(M) \) is the subgroup of \( \text{Diff}(M) \) consisting of diffeomorphisms which are the identity outside of some compact set and \( \mathfrak{D}_0(M) \) is the arc component of \( i_M \), the identity map of \( M \), in \( \mathfrak{D}(M) \). If \( M \) is compact \( \mathfrak{D}(M) \) is locally arcwise connected and \( \mathfrak{D}_0(M) \) is open in \( \mathfrak{D}(M) \) and in fact in \( \mathfrak{M}(M, M) \). For a definition of the \( C^K \)-topology and a proof of the statements made above, see [6]. If \( G \) is a Lie group we denote by \( \mathfrak{G}(G, M) \) the space of differentiable actions of \( G \) on \( M \), i.e. continuous homomorphisms of \( G \) into \( \text{Diff}(M) \), topologized with the compact-open topology. If \( \phi: g \mapsto g^* \) is an element of \( \mathfrak{G}(G, M) \) then by a theorem of D. Montgomery [2] \( \phi: (g, m) \mapsto g^*m \) is an element of \( \mathfrak{M}(G \times M, M) \). Given \( \phi \in \mathfrak{G}(G, M) \) and \( f \in \text{Diff}(M) \) then \( \phi \) composed with the inner automorphism of \( \text{Diff}(M) \) defined by \( f \) is another element \( f\phi \) of \( \mathfrak{G}(G, M) \)(\( g^* = fg^*f^{-1} \)). Clearly \( (f, \phi) \mapsto f\phi \) is jointly continuous\(^2\) and defines an action of \( \text{Diff}(M) \) on \( \mathfrak{G}(G, M) \). We henceforth consider \( \mathfrak{G}(G, M) \) as a \( \text{Diff}(M) \)-space and, \textit{a fortiori} as a \( \mathfrak{D}(M) \) and \( \mathfrak{D}_0(M) \)-space. Note that the orbit space \( \mathfrak{G}(G, M)/\text{Diff}(M) \) is just the set of equivalence classes of actions of \( G \) on \( M \).

2. Statement of main theorem and consequences. The following theorem will be proved in §3.

**Theorem A.** If \( M \) is a compact differentiable manifold and \( G \) is a compact Lie group then the \( \mathfrak{D}_0(M) \)-space \( \mathfrak{G}(G, M) \) admits local cross sections; i.e. given \( \phi_0 \in \mathfrak{G}(G, M) \) there is a neighborhood \( U \) of \( \phi_0 \) in...
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\[ \alpha(G, M) \text{ and a continuous map } \chi: U \to \mathcal{D}_0(M) \text{ such that } \chi(\phi_0) = i_M \text{ and } \chi(\phi)\phi_0 = \phi. \]

**Corollary 1.** If \( \phi_1 \) is a continuous arc in \( \alpha(G, M) \) then there is a continuous arc \( f_1 \) in \( \mathcal{D}_0(M) \) such that \( f_0 = i_M \) and \( \phi_1 = f_1\phi_0. \)

**Remarks.** Corollary 1 was proved in [7] by the author and T. E. Stewart under the added hypothesis that \( (g, m, t) \to \phi_t(g, m) \) was jointly differentiable in all three variables. It was shown there by counter-example that Corollary 1 is invalid if we consider continuous rather than differentiable actions or if we drop either of the conditions that \( G \) or \( M \) be compact. It follows that all these conditions are also necessary for the validity of Theorem A.

Using that \( \mathcal{D}_0(M) \) is locally arcwise connected:

**Corollary 2.** \( \alpha(G, M) \) is locally arcwise connected. If \( \phi_0 \in \alpha(G, M) \) then its orbit under \( \mathcal{D}_0(M) \) is its arc component in \( \alpha(G, M) \) hence an open set, and its orbit under \( \mathcal{D}(M) \) (i.e. the class of actions equivalent to \( \phi_0 \)) is also open and so a union of arc components. Moreover if \( \Delta = \{ f \in \mathcal{D}(M) | f\phi_0 = \phi_0 \} \) is the group of automorphisms of the differentiable \( G \)-space \( (M, \phi_0) \) then \( f\Delta \to f\phi_0 \) is a homeomorphism of \( \mathcal{D}(M)/\Delta \) onto \( \mathcal{D}(M)/\phi_0. \)

Since \( \alpha(G, M) \) is separable metric and each equivalence class is open:

**Corollary 3.** There are at most countably many inequivalent differentiable actions of \( G \) on \( M. \)

**Remarks.** It seems likely that by modifying a construction of R. Bing [1] one could construct uncountably many continuous actions of \( Z_2 \) on \( S^3 \) with fixed point sets pairwise inequivalently embedded 2-spheres. These actions would of course all be inequivalent.

The following extension theorem generalizes Theorem A. On the other hand it is an easy consequence of Theorem A above and Theorem B of [6].

**Theorem B.** Let \( H \) be a Lie group, \( W \) a differentiable manifold (neither necessarily compact), \( G \) a compact subgroup of \( H \), and \( M \) a compact submanifold of \( W \). Let \( \psi_0 \in \alpha(H, W) \) such that \( M \) is invariant under \( \psi_0|G \) and let \( \phi_0 \in \alpha(G, M) \) be the induced action of \( G \) on \( M \). Then given any neighborhood \( \theta \) of \( M \) in \( W \) there exists a neighborhood \( U \) of \( \phi_0 \) in \( \alpha(G, M) \) and a map \( \psi: U \to \alpha(H, W) \) such that \( \psi(\phi_0) = \psi_0, \psi(\phi)|G \) leaves \( M \) invariant and induces \( \phi \) on \( M \), and \( \psi(\phi) \) agrees with \( \psi_0 \) outside \( \theta \). In fact there is a continuous map \( \chi: U \to \mathcal{D}_0(W) \) such that \( \chi(\phi) \) is the identity outside \( \theta \) and such that \( \psi(\phi) = \chi(\phi)\psi_0 \) satisfies the above conditions.
3. **Proof of Theorem A.** By a theorem proved independently by the author [5] and G. D. Mostow [4] there exists an orthogonal representation $g \mapsto g^\psi$ of $G$ in a Euclidean vector space $V$ and a differentiable $\phi_0$-equivariant embedding $f_0: M \to V$. Let $\theta$ be a tubular neighborhood of $f_0(M)$ in $V$ with respect to the Euclidean metric. Then $\theta$ is invariant under the representation $\psi$ and the map $\pi: \theta \to f_0(M)$ carrying a point of $\theta$ into the unique nearest point of $f_0(M)$ is a differentiable equivariant retraction of $\theta$ onto $f_0(M)$. Given $\phi \in \mathcal{G}(G, M)$ define $f_\phi: M \to V$ by $f_\phi(m) = \int g^{-1} f_0(g^\phi m) \, dg$ where the integral is with respect to Haar measure on $G$. Then (cf. [4, p. 434]) $f_\phi$ is $\phi$-equivariant and clearly $f_{\phi_0} = f_0$. The map $F_\phi \in \mathfrak{M}(G \times M, V)$ defined by $F_\phi(g, m) = \psi(g^{-1}, f_0 \circ \phi(g, m))$ is easily seen to depend continuously on $\phi \in \mathcal{G}(G, M)$ and since $f_\phi = \int F_\phi(g, m) \, dg$ it follows that $\phi \mapsto f_\phi$ is a continuous map of $\mathcal{G}(G, M)$ into $\mathfrak{M}(M, V)$. Then for $\phi$ in a neighborhood $U$ of $\phi_0$ in $\mathcal{G}(G, M)$ $f_\phi(M) \subseteq \Theta$ so $\sigma(\phi) = f_{\phi_0}^{-1} \circ \pi \circ f_\phi \in \mathfrak{M}(M, M)$. Now $\sigma: U \to \mathfrak{M}(M, M)$ is continuous and clearly $\sigma(\phi_0) = \text{id}_M$. Since $\mathfrak{D}(M)$ is open in $\mathfrak{M}(M, M)$, for some smaller neighborhood $U$ of $\phi_0$ in $\mathcal{G}(G, M)$ $\sigma: U \to \mathfrak{D}(M)$. Since $f_\phi, \pi$, at $f_{\phi_0}$ are respectively $\phi$, $\pi$, and $\phi_0$-equivariant maps into $(V, \psi)$ it follows that $\sigma(\phi)g^\psi = g^\psi \sigma(\phi)$ or putting $\chi(\phi) = \sigma(\phi)^{-1}$, $\chi(\phi)\phi_0 = \phi$. Q.E.D.

4. **Conjugacy of neighboring compact subgroups of $\text{Diff}(M)$.** It is suggested by Theorem A that an analogue of the Montgomery and Zippin conjugacy theorem for neighboring compact subgroups of a Lie group [3] might hold for $\text{Diff}(M)$, i.e. that given a compact subgroup $G$ of $\text{Diff}(M)$ every compact subgroup of $\text{Diff}(M)$ sufficiently close to $G$ is conjugate in $\text{Diff}(M)$ to a subgroup of $G$. This in fact is the case and was the basis of an earlier more complicated proof of Theorem A. A proof will appear elsewhere.

**References**


**Brandeis University**