A MINIMAL DEGREE LESS THAN 0'

BY GERALD E. SACKS

Communicated by A. W. Tucker, April 13, 1961

Clifford Spector in [4] proved that there exists a minimal degree less than 0'. J. R. Shoenfield in [3] asked: "Does there exist a minimal degree \( a \) such that \( 0^m < a \)?" We show that the answer to his question is yes! Our notation is that of [4].

We say that \( b \) strictly extends \( a \) if \( b \) and \( a \) are distinct sequence numbers, and if the sequence represented by \( b \) extends the one represented by \( a \); we express this symbolically as \( \text{SExt} (b, a) \). If \( \{a_0, a_1, a_2, \ldots \} \) is a sequence of sequence numbers such that for each \( i \), \( a_{i+1} \) strictly extends \( a_i \), then there is a unique function \( f(n) \) such that for each \( i \) there is an \( m \) with the property that \( f(m) = a_i \); if \( \{a_0, a_1, a_2, \ldots \} \subseteq S \), then we say \( f(n) \) is a function associated with \( S \). Spector in [4] obtained a function of minimal degree as the unique function associated with every member of a contracting sequence of sets of sequence numbers. Our construction is inspired by his, but it differs markedly from his in one respect: each one of our sets of sequence numbers will be recursively enumerable, whereas each one of his was recursive.

For each natural number \( c \), let \( c^* \) be the unique, recursively enumerable set which has \( c \) as a Gödel number. There exists a recursive function \( g(n) \) such that for each \( c \), \( g(c) \) is the Gödel number of the representing function of a recursive predicate \( R_c(m, x) \) with the property that \( x \in c^* \) if and only if \( (\exists m) R_c(m, x) \). We define a recursive predicate \( H(c, t, e, x, m, b, d) \) which is basic to our construction:

\[
H(c, t, e, x, m, b, d) = (i) < (\text{SExt}(t, i), R_c(m, x)) & \text{ and } T^1((x)_i, e, b, (d)_i) & \text{ and } U((d)_0) \neq U((d)_1).
\]

We define a partial recursive function \( Y(c, t, e) \):

\[
Y(c, t, e) = \begin{cases} 
\mu x H(c, t, e, (x)_0, (x)_1, (x)_2, (x)_3) & \text{if } (\exists x) H(c, t, e, (x)_0, (x)_1, (x)_2, (x)_3) \\
\text{undefined otherwise.} & 
\end{cases}
\]

We define a recursively enumerable set of sequence numbers denoted by \( W(c, t, e) \): (a) \( t \in W(c, t, e) \) if \( t \) is a sequence number; (b) if \( u \in W(c, t, e) \) and if \( Y(c, u, e) \) is defined, then \( (Y(c, u, e))_0, (Y(c, u, e))_1, (Y(c, u, e))_2 \) and \( (Y(c, u, e))_3 \) are in \( W(c, t, e) \); and (c) every member of \( W(c, t, e) \) is

\[\text{The author is a predoctoral National Science Foundation Fellow.}\]
obtained by an application of (a) followed by finitely many applications of (b). It is clear that there exists an effective procedure for computing a Gödel number of $W(c, t, e)$ from the triple $(c, t, e)$. We define the recursive function $V(c, t, e)$ to be that function whose value for the triple $(c, t, e)$ is equal to the result of applying this effective procedure to the triple $(c, t, e)$.

We are ready to define four functions simultaneously by induction, $Q(i, j), v(i), u(i)$ and $t(i)$, where $i$ and $j$ are natural numbers. $Q(i, j)$ will take only $0$ and $1$ as values. The sequence $\{u(0), u(1), u(2), \cdots\}$ will consist of sequence numbers such that for each $n$, $u(n+1)$ will strictly extend $u(n)$; the unique function $h(n)$ associated with this sequence will have minimal degree.

Let $q$ be a Gödel number of the set of all sequence numbers. Let $0(n)$ be the function which is everywhere $0$. If $t$ is a sequence number, let $w(t)$ be the least $x$ such that $SExt((x)_0, t), SExt((x)_1, t), (x)_0$ does not strictly extend $(x)_1, (x)_1$ does not strictly extend $(x)_0$ and $(x)_0 \neq (x)_i$. We set $t(0) = 0$ and $Q(i, 0) = 1$ for all $i > 0$. We set $u(0) = 2^{1+\#(0)}$ if the latter expression is defined; otherwise we set $u(0) = 2$. If $Y(q, u(0), 0)$ is defined, then we set $Q(0, 0) = 1$ and $v(0) = (Y(q, u(0), 0))_0$; otherwise we set $Q(0, 0) = 0$ and $v(0) = v(u(0))$.

Now suppose that $Q(i, s-1)$ has been defined for all $i$, and that $v(s-1)$ and $u(s-1)$ have also been defined, where $s > 0$. Suppose further that $(v(s-1))_0$ and $(v(s-1))_1$ are distinct sequence numbers such that neither one strictly extends the other. Let $u(s)$ be the least one of $(v(s-1))_0$ and $(v(s-1))_1$, which is not strictly extended by $\prod_{i<s} b_i^{1+(u(i))_0}$, if the latter expression is defined; otherwise, let $u(s) = (v(s-1))_0$. Let $\{i \mid i < s, Q(i, s - 1) = 1\} \cup \{s, s + 1\} = \{i_1, i_2, \cdots, i_{r+1}\}$, where $i_1 < i_2 < \cdots < i_{r+1}$. Let $v_0 = q$; and for each $k < r_s$, let $v^{k+1} = Y(v_k, u(s), i_{k+1})$. Let $t(s)$ be $r_s + 1$ if $Y(v_k, u(s), i_{k+1})$ is defined for all $k < r_s$; otherwise, let $t(s)$ be the least $k \leq r_s$ for which $Y(v_k, u(s), i_k)$ is not defined. We define $v(s)$ and $Q(i, s)$ for all $i$:

$$v(s) = \begin{cases} w(u(s)) & \text{if } t(s) = 1, \\ (Y(v_{t(s)-1}, u(s), i_{t(s)}))_0 & \text{if } t(s) = k + 1 > 1. \end{cases}$$

$$Q(i, s) = \begin{cases} Q(i, s - 1) & \text{if } i < i_{t(s)}, \\ 0 & \text{if } i = i_{t(s)} \leq s, \\ 1 & \text{if } i > i_{t(s)} \text{ or if } i > s. \end{cases}$$

Let $h(n)$ be the unique function associated with the sequence $\{u(0), u(1), u(2), \cdots\}$ of sequence numbers. It is clear from the definition of $u(s)$ that $h(n)$ is nonrecursive. To see that $h(n)$ has degree...
less than or equal to 0', observe that for each fixed \( s > 0 \), \( u(s) \) can be computed if the value of \( v(s-1) \) and finitely many truth-values of \( (Ey)T(0(y), s, x, y) \) are known, \( t(s) \) can be computed if the values of \( u(s), Q(0, s-1), Q(1, s-1), \cdots, Q(s-1, s-1) \) and finitely many truth-values of \( (Ey)T(e, x, y) \) are known, and both \( v(s) \) and \( Q(i, s) \) for all \( i \) can be computed if the values of \( u(s) \) and \( t(s) \), \( Q(0, s-1), Q(1, s-1), \cdots, Q(s-1, s-1) \) are known.

We now show by induction on \( i \) that for each \( i \) there is an \( s^{**} \) such that \( Q(i, s-1) = Q(i, s) \) for all \( s \geq s^{**} \). Suppose this is so for all \( i < k \). Let \( s^* \) be such that \( Q(i, s-1) = Q(i, s) \) for all \( i < k \) and all \( s \geq s^* \). Suppose (for the sake of a reductio ad absurdum) that \( s' \geq s^* \) and \( Q(k, s') = 1 \). It follows from the definition of \( Q(k, s') \) that \( 0 \leq i_{s'(v)}, k, Q(i_{s(v)}, s'-1) = 1 \) and \( Q(i_{s(v)}, s') = 0 \). But this last is impossible because either \( k = 0 \) or \( s' \geq s^* \). It must be the case that there is an \( s^{**} \) such that \( Q(k, s'-1) = Q(k, s') \) for all \( s' \geq s^{**} \). For each \( i \), let \( s(i) \) be the least \( s \) such that \( Q(i, s'-1) = Q(i, s') \) for all \( s' \geq s \). It can be shown that the function \( s(i) \) is not recursive.

We define a contracting sequence of sets of sequence numbers. We set \( F_0 \) equal to the recursively enumerable set which has \( V(q, u(s(0))), 0 \) as a Gödel number if \( Q(0, s(0)-1) = 1 \), and equal to \( \{ s \mid Ext(s, u(s(0))) \} \) otherwise. For each \( j > 0 \), let \( f_{j-1} \) be a Gödel number of \( F_{j-1} \). We set \( F_j \) equal to the recursively enumerable set which has \( F(j-1), u(s(j)), j \) as a Gödel number if \( Q(j, s(j)-1) = Q(j, s') = 1 \), and equal to \( \{ s \mid Ext(s, u(s(j))), s \in F_{j-1} \} \) otherwise.

Suppose that \( \{ e \}^k(n) \) is defined for all \( n \). We claim that either \( \{ e \}^k(n) \) is recursive or \( h(n) \) is recursive in \( \{ e \}^k(n) \). Suppose that \( Q(e, s(e)-1) = 0 \), then \( \{ e \}^k(n) \) is recursive. This is so, because for each \( n \), there is an \( s \in F_e \) and a \( d \) such that \( T^4(s, e, n, d) \), and because for each such \( s \) and \( d \), \( U(d) = \{ e \}^k(n) \). Suppose that \( Q(e, s(e)-1) = 1 \), then \( h(n) \) is recursive in \( \{ e \}^k(n) \). This is so because there is only one function \( w(n) \) associated with \( F_e \) such that \( \{ e \}^w(n) = \{ e \}^k(n) \) for all \( n \). To compute \( h(n) \) from \( \{ e \}^k(n) \), we merely simultaneously enumerate \( F_e \) and the set of all deductions; whenever a choice has to be made between two sequence numbers, \( s_1 \) and \( s_2 \), of \( F_e \), only one of which, let us say \( s_2 \), represents an initial segment of \( h(n) \), there is nothing to fear because eventually some deduction will make clear that \( (Ed, b)(T^4(s_1, e, b, (d)o)\&U(((d)o) \neq \{ e \}^k(b)\&T^4(s_2, e, b, (d)o)\&U(((d)o) = \{ e \}^k(b)) \).

This completes the proof of Theorem 1 below. By making inessential changes Theorem 2 is proved.

**Theorem 1.** There exists a minimal degree less than \( 0' \).
Theorem 2. For each degree c, there is a degree g greater than c and less than c' such that c < b < g for no degree b.

References


Cornell University