ON A PROBLEM OF P. A. SMITH

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Communicated by Deane Montgomery, March 31, 1961

1. Introduction. Throughout this note, $\mathbb{Z}_2$ denotes the group of integers mod 2 and cohomology means the Alexander-Wallace-Spanier cohomology with coefficients in $\mathbb{Z}_2$. By a cohomology projective $n$-space we mean a compact Hausdorff space $Y$ whose cohomology ring $H^*(Y)$ is isomorphic to that of the real projective $n$-space. In [2], Smith proved that if $\mathbb{Z}_2$ acts effectively on the real projective $n$-space such that the fixed point set $F(\mathbb{Z}_2)$ is nonempty, then $F(\mathbb{Z}_2)$ has exactly two components $A_1$ and $A_2$, where $A_i$ is a cohomology projective $n_i$-space $(i=1, 2)$ and $n_1+n_2=n-1$. Smith then asked whether the result is true if the real projective $n$-space is replaced by a cohomology projective $n$-space. The purpose of this note is to give a positive answer to the question.

We wish to point out that the inclusion of ring structure in the definition of a cohomology projective $n$-space is indispensable as we may see from the following example. Let $Y$ be the one-point union of a 1-sphere $S^1$ and a 2-sphere $S^2$. Clearly $H^*(Y)$ as a group is the same as the cohomology group of a projective plane. Let $T$ be a generator of $\mathbb{Z}_2$ and define the action of $T$ on $Y$ such that on $S^1$ it is the reflexion with respect to the diameter passing through the point of contact. Then the fixed point set consists of three isolated points.

2. A construction. The proof of Smith's theorem in [2] has used the fact that a projective $n$-space admits an $n$-sphere as its two-folded covering space. It is therefore quite natural to expect that a cohomology projective $n$-space $Y$ admits a cohomology $n$-sphere as its two-folded covering space. In the following we give a construction of such a cohomology $n$-sphere which is very similar to the construction of a covering space of a pathwise connected, locally pathwise connected, and locally pathwise simply connected space, with the dual of $H^1(Y)$ playing the role of fundamental group.

Let $Y$ be a connected compact Hausdorff space and let $\alpha \in H^1(Y)$ be a nonzero element. Let $f: Y^2 \to \mathbb{Z}_2$ be a 1-cocycle representing $\alpha$; then there exists an open covering $\mathcal{U}$ of $Y$ such that

$$f(y_0, y_2) = f(y_0, y_1) + f(y_1, y_2) \quad \text{whenever} \quad y_0, y_1, y_2 \in V \in \mathcal{U}.$$
Fix a point \( b \in Y \). By a \( \mathcal{V} \)-chain with base point \( b \) we mean a finite sequence \( (y_i)_{i=0}^n \) of points of \( Y \) such that \( y_0 = b \) and \( \{y_{i-1}, y_i\} \) is contained in some \( V \in \mathcal{V} \) for all \( i = 1, 2, \ldots, n \), the set of all \( \mathcal{V} \)-chains with base point \( b \) is denoted by \( \mathfrak{X} \). Two \( \mathcal{V} \)-chains \( (y_i)_{i=0}^n \) and \( (y'_i)_{i=0}^m \) are said to be equivalent if

\[
\begin{align*}
(i) \quad & y_n = y'_m, \\
(ii) \quad & \sum_{i=1}^{n} f(y_{i-1}, y_i) = \sum_{j=1}^{m} f(y'_{j-1}, y'_j).
\end{align*}
\]

The quotient set of \( \mathfrak{X} \) under this equivalence relation is denoted by \( X \) and the equivalence class of \( (y_i)_{i=0}^n \) is denoted by \( [y_i]_{i=0}^n \).

Now we topologize \( X \) as follows. Let \( x = [y_i]_{i=0}^n \in X \) and \( \mathcal{B}(y_n) \) be a base of neighborhood of \( y_n \) such that every \( B(y_n) \in \mathcal{G}(y_n) \) is contained in some \( V \in \mathcal{V} \). To each \( B(y_n) \in \mathcal{G}(y_n) \), we define

\[
\beta(x) = \left\{ [y_i']_{i=0}^m \mid y_m' \in B(y_n), \sum_{i=1}^{m} f(y_{i-1}, y_i) + f(y_n, y_m') + \sum_{j=1}^{m} f(y'_{j-1}, y'_j) = 0 \right\}.
\]

It is easily verified that \( X \) is made a Hausdorff space with

\[
\mathcal{G}(x) = \{ B^*(x) \mid B(y_n) \in \mathcal{G}(y_n) \}
\]

as a base of neighborhoods of \( x \).

Define a map \( \pi: X \rightarrow Y \) by \( \pi([y_i]_{i=0}^n) = y_n \), it is straightforward to verify that \( \pi \) is well-defined and is a local homeomorphism of \( X \) onto \( Y \).

Obviously, to each \( y \in Y \), \( \pi^{-1}(y) \) has at most two points. We now claim that it has exactly two points. To see this, it suffices to consider the case when \( y = b \). Since \([b]\) is one point of \( \pi^{-1}(b) \), all we have to do is to exhibit a \( \mathcal{V} \)-chain \( (y_i)_{i=0}^n \) with \( y_0 = y_n = b \) and \( \sum_{i=1}^{n} f(y_{i-1}, y_i) = 1 \). Suppose such a chain does not exist, then we can define a 0-cochain \( g: Y \rightarrow \mathbb{Z}_2 \) by \( g(y) = \sum_{i=1}^{n} f(y_{i-1}, y_i) \), where \( (y_i)_{i=0}^n \) is any \( \mathcal{V} \)-chain with base point \( b \) with \( y_n = y \). Such a chain exists in view of the connectedness of \( Y \) and \( g \) is clearly well-defined. Now if \( \{y, y'\} \in V \in \mathcal{V} \), we have

\[
g(y') - g(y) = \sum_{i=1}^{n} f(y_{i-1}, y_i) + f(y, y') - \sum_{i=1}^{n} f(y_{i-1}, y_i) = f(y, y').
\]

But this means \( f - \delta g \) has empty support, contradicting the assumption that \( \alpha \neq 0 \).

Now let \( T \) be the generator of \( \mathbb{Z}_2 \) and define the action of \( T \) by exchanging the two points in \( \pi^{-1}(y) \) for each \( y \in Y \). We clearly obtain
a free action of \( Z_2 \) on the compact Hausdorff space \( X \) with \( Y = X/Z_2 \).

Define a 0-cochain \( h: X \to Z_2 \) by

\[
h([y], y_0) = \sum_{i=1}^{n} f(y_{i-1}, y_i).
\]

A similar argument as above shows that \( \pi^*(\alpha) \) is the cohomology class of \( \delta h \).

Suppose that now \( Y \) is a cohomology projective \( n \)-space and that \( \alpha \) is the generator of the cohomology ring \( H^*(Y) \). We claim that \( X \) is a cohomology \( n \)-sphere. As seen in [1], we have the exact Smith-Gysin sequence

\[
\cdots \to H^k(Y) \xrightarrow{\pi^*} H^k(X) \xrightarrow{\tau^*} H^{k+1}(Y) \to \cdots.
\]

Since \( \pi^*(\alpha) = 0 \) and \( \pi^* \) is a ring homomorphism, it follows that \( \pi^*: H^k(Y) \to H^k(X) \) is trivial for all \( k > 0 \). This is enough to conclude that

\[
H^k(X) = \begin{cases} Z_2, & k = 0, n, \\ 0, & \text{otherwise.} \end{cases}
\]

3. **Main theorem.**

**Theorem.** If \( Z_2 \) acts effectively on a cohomology projective \( n \)-space \( Y \) such that the fixed point set \( F(Z_2) \) is nonempty, then \( F(Z_2) \) has exactly two components \( A_1 \) and \( A_2 \) where each \( A_i \) is a cohomology projective \( n \)-space \((i = 1, 2)\) and \( n_1 + n_2 = n - 1 \).

**Proof.** Let \( S \) be the generator of \( Z_2 \). In the construction of \( X \) given in the last section, we may choose the base point \( b \) in \( F(Z_2) \) and we may assume that \( \mathcal{U} \) is \( S \)-invariant (i.e. \( \mathcal{U}(V) \subseteq \mathcal{U} \) for all \( V \subseteq \mathcal{U} \)). It follows that \( S \) maps \( \mathcal{U} \)-chains with base point \( b \) into themselves or \( S \) induces a transformation on \( \mathcal{X} \). Observe that \( S \) also induces an automorphism \( S^* \) on \( H^1(Y) \); hence we must have \( S^*(\alpha) = \alpha \). It is easily seen that this fact implies that \( S \) maps equivalent \( \mathcal{U} \)-chains into themselves, in other words \( S \) induces a transformation \( \hat{S} \) on the space \( X \) which is clearly compatible with \( \pi \) (i.e. \( \pi \circ \hat{S} = S \circ \pi \)). This means we have an action of the group \( Z_2 \times Z_2 \) on a cohomology \( n \)-sphere \( X \). The rest of the proof is word by word the same as given in [2].

**References**


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