A REPRESENTATION OF THE INFINITESIMAL GENERATOR OF A DIFFUSION PROCESS

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0. Introduction. Let \( \Omega \) be a connected locally compact metric space and let \( C(\Omega) \) denote, as usual, the Banach space of bounded continuous real functions on \( \Omega \). A diffusion process (see [1] for definitions) is a semi-group \( \{ T_t; t > 0 \} \) of positivity preserving bounded linear transformations on \( C(\Omega) \) which is strongly continuous for \( t > 0 \). Such semi-groups are also required to be of local character; i.e., if \( x \) vanishes in a neighborhood of a point \( \xi \in \Omega \), then

\[
Ax(\xi) = \lim_{t \to 0} \frac{T_t x - x}{t} (\xi) = 0.
\]

Consider an arbitrary element \( x \in C(\Omega) \). If (i) the above limit exists for all \( \eta \) in a neighborhood \( W \) of \( \xi \), (ii) the convergence is bounded on this neighborhood, and (iii) \( Ax \) is continuous on \( W \), then \( x \) is said to be in the local domain of the operator \( A \) at \( \xi \). \( x \) is said to be in the global domain, \( D(A) \), of the operator \( A \) whenever \( W = \Omega \). Feller (see [1]) has posed the problem of characterizing the operator \( A \). A local representation of such operators will be discussed in this note.

One of the essential properties of the operator \( A \) is the maximum property; i.e., \( Ax(\xi) \leq 0 \) whenever \( x \) is in the local domain of \( A \) at \( \xi \) and \( x \) has a null maximum at \( \xi \). Before discussing the representation of \( A \), a few remarks concerning the denseness in \( C(\Omega) \) of the global domain of the operator \( A \) are in order. A null point of \( A \) is a point \( \xi \in \Omega \) such that \( Ax(\xi) = \xi(\xi) = 0 \) for all \( x \) in the local domain of \( A \) at \( \xi \). Feller has shown that the set \( N \) of null points is a closed set. He has also shown that if \( x \) vanishes outside a compact set which does not meet \( N \), then there is a sequence \( X_\lambda \) in the global domain of \( A \) such that \( X_\lambda \to x \) strongly [1]. Using this result, one can show that \( D(A) \) is locally dense in \( C(\Omega) \) at each point \( \xi \in \Omega - N \); i.e., if (i) \( \xi \in \Omega - N \), (ii) \( W \) is a neighborhood of \( \xi \) such that \( W \subset \Omega - N \), and (iii) \( x \in C(\Omega) \), then \( x \) can be approximated uniformly over \( W \) by an element of \( D(A) \).

Another type of point at which the operator \( A \) may be degenerate is the absorption point; i.e., \( \xi \) is an absorption point if there is a real

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number \( c \) such that \( Ax(\xi) = cx(\xi) \) for all \( x \) in the local domain of \( A \) at \( \xi \). It will be assumed throughout this note that \( \xi \) is a fixed point of \( \Omega \) which is neither a null point nor an absorption point. This assumption implies that there is an \( x \) in the local domain of \( A \) at \( \xi \) such that \( Ax(\xi) > 0 \); \( V \) will denote a neighborhood of \( \xi \) with compact closure such that \( \overline{V} \cap N = \emptyset \) and \( Ax > 0 \) on \( \overline{V} \). If \( \eta \in V \), then \( D^*(A, \eta) \) will denote the functions in the local domain of \( A \) at \( \eta \) restricted to \( \overline{V} \). \( D^*(A) \) will denote the set of functions obtained by restricting elements of \( D(A) \) to \( \overline{V} \), and \( C(\overline{V}) \) will denote the Banach space of continuous real functions on \( \overline{V} \). By the above remarks, \( D^*(A) \) is dense in \( C(\overline{V}) \).

1. Generalized harmonic measures. It will be assumed in this section that \( 1 \in D(A) \) and \( A1 = 0 \). Let \( U \) be an open subset of \( V \). The boundary of \( U \) will be denoted by \( U' \). A function \( y \) in \( D^*(A, \eta) \) for all \( \eta \in U \) will be called subregular (superregular) on \( U \) if \( Ay \leq 0 \) (\( Ay \geq 0 \)) on \( U \). A function is regular on \( U \) if it is both subregular and superregular on \( U \). The set \( P \) of functions subregular on \( U \) has the following properties:

(a) \( P \) is a wedge in \( C(\overline{V}) \); i.e., \( P \) is a convex set in \( C(\overline{V}) \) and \( tP \subseteq P \) for \( t \geq 0 \).

(b) \( x \in P \) implies \( x(\eta) \leq \sup_U x \) for all \( \eta \in U \).

(c) \( P \) contains a nonzero element.

The second assertion follows from the fact that \( A1 = 0 \) and that \( A \) possesses the maximum property. These three properties suffice to prove the following theorem.

**Theorem 1.** For each \( \eta \in U \), there is a regular Borel measure \( \rho(\eta, U, \cdot) \) defined on the Borel subsets of \( U' \) such that

\[
x(\eta) \leq \int_{U'} x(\sigma) \rho(\eta, U, d\sigma)
\]

for each \( x \) subregular on \( U \). Moreover, \( \rho(\eta, U, U') = 1 \).

**Sketch of Proof.** Consider \( C(\overline{V}) \times \mathbb{R} \), the Cartesian product of \( C(\overline{V}) \) with the set of real numbers. The set \( E \) of all pairs \( (x, \alpha) \) where \( x \in C(\overline{V}) \) and \( \alpha \geq \sup_U x \) is a convex body in this product space. The set \( F \) of all pairs \( (x, x(\eta)) \) where \( x \) is subregular on \( U \) is a convex set in the product space. Moreover, \( \text{Int } E \cap F = \emptyset \). Using the Eidelheit separation theorem, there is a linear functional \( y^* \) on \( C(\overline{V}) \) such that \( x(\eta) \leq y^*(x) \) for all \( x \) subregular on \( U \) and \( y^*(x) \leq \sup_U x \) for all \( x \in C(\overline{V}) \). The Riesz representation theorem can be used to represent the positivity preserving linear functional \( y^* \) as a measure on \( \overline{V} \). To show that this measure is concentrated on \( U' \), it need only be ob-
served that \( y^*(z) = 0 \) for any \( z \in C(\overline{V}) \) which is zero on \( U' \) and strictly positive elsewhere.

A measure of the type described in the preceding theorem will be called a generalized harmonic measure. The integral relative to a generalized harmonic measure will be denoted by \( L(\eta, U, \cdot) \).

### 2. Representations

For the time being, it will be assumed that \( 1 \in D(A) \) and \( A1 = 0 \). Again \( U \) will be an open subset of \( V \). The inequality of the following lemma is the starting point of the representation.

**Lemma 2.** There is an \( x \in D^*(A) \) such that \( Ax > 0 \) on \( \overline{V} \), \( x(\eta) < L(\eta, U, x) \) for all \( \eta \in U \), and

\[
\inf_{\overline{V}} \frac{Ax}{Ax} \leq \frac{L(\eta, U, z) - z(\eta)}{L(\eta, U, x) - x(\eta)} \leq \sup_{\overline{V}} \frac{Ax}{Ax}
\]

whenever \( Az \) is defined on \( \overline{U} \).

**Sketch of Proof.** One first shows that there is a \( y \) (which may depend upon \( \eta \) and \( U \)) such that \( y(\eta) < L(\eta, U, y) \) and \( Ay > 0 \) on \( \overline{U} \) as follows. Suppose the contrary; i.e., \( y(\eta) = L(\eta, U, y) \) for all \( y \) such that \( Ay > 0 \) on \( \overline{V} \). By assumption, there is an \( x \) such that \( Ax > 0 \) on \( \overline{V} \). Consider any \( y \) such that \( Ay \) is defined on \( \overline{U} \). Each such \( y \) can be represented in the form \( z - tx \) where \( Az > 0 \) on \( \overline{U} \) and \( t \) is sufficiently large. It follows that \( y(\eta) = L(\eta, U, y) \) for all \( y \) for which \( Ay \) is defined on \( \overline{U} \). But since the class of such functions is dense in \( C(\overline{U}) \), the evaluation linear functional \( y^*(y) = y(\eta) \) and the linear functional \( L(\eta, U, \cdot) \) are equal. This, however, is not possible. This proves that there is a \( y \) such that \( Ay > 0 \) on \( \overline{U} \) and \( y(\eta) < L(\eta, U, y) \).

A preliminary version of the lemma is now obtained as follows. Consider any \( z \) such that \( Az \) is defined on \( \overline{U} \). For \( t \geq -\inf_{\overline{U}}(Az/Ay) \), \( A(z + ty) \geq 0 \) on \( \overline{U} \). By Theorem 1, \( z(\eta) + ty(\eta) \leq L(\eta, U, z + ty) \). Re-arranging terms and letting \( t \) approach \( -\inf_{\overline{U}}(Az/Ay) \) results in the left inequality (with \( x \) replaced by \( y \)). The other inequality is proved similarly. Having proved the inequality with \( x \) replaced by \( y \), it follows that \( z(\eta) < L(\eta, U, z) \) for any \( z \) such that \( Az > 0 \) on \( \overline{V} \) and that \( y \), which may depend upon \( \eta \) and \( U \), may be replaced by any such \( z \).

In passing it is worth noting that the preceding inequality can be used to show that every generalized second order differential operator on \( C(\Omega) \) as herein considered has a closed extension. The following theorem is an obvious consequence of Lemma 2.
Theorem 3. There is an \( x \in D^*(A, \xi) \) such that for each \( \eta \in V \) and each \( z \in D^*(A, \eta) \)
\[
Az(\eta) = Ax(\eta) \lim_{U \downarrow \eta} \frac{L(\eta, U, z) - z(\eta)}{L(\eta, U, x) - x(\eta)}.
\]

The requirement that \( 1 \in D(A) \) and \( A1 = 0 \) can now be removed.

Theorem 4. There is a neighborhood \( W \) of \( \xi \), a function \( x \in D^*(A, \xi) \) with \( x > 0 \) on \( \overline{W} \), and a function \( y \in D^*(A, \xi) \) with \( xAy - yAx > 0 \) on \( \overline{W} \) such that for each \( \eta \in W \) and each \( z \in D^*(A, \eta) \)
\[
Az(\eta) = \frac{z(\eta)}{x(\eta)} Ax(\eta) + \frac{x(\eta) A y(\eta) - y(\eta) A x(\eta)}{x(\eta)} \lim_{U \downarrow \eta} \frac{L(\eta, U, z/x) - z(\eta)/x(\eta)}{L(\eta, U, y/x) - y(\eta)/x(\eta)}.
\]

Sketch of Proof. Since \( D(A) \) is locally dense at \( \xi \), there is an \( x \in D^*(A, \xi) \) such that \( x(\xi) > 0 \). Choose a neighborhood \( W_1 \subset V \) of \( \xi \) such that \( x > 0 \) on \( \overline{W_1} \). Since \( \xi \) is neither a null point nor an absorption point, there is a \( y \in D^*(A, \xi) \) such that \( x(\xi) A y(\xi) - y(\xi) A x(\xi) > 0 \). Choose a neighborhood \( W \) of \( \xi \) such that \( W \subset W_1 \) and \( xAy - yAx > 0 \) on \( \overline{W} \). After restricting all functions to \( \overline{W} \), one defines an operator \( B \) on quotients of the form \( z/x \), where \( z \in D^*(A, \eta) \) and \( \eta \in W \), by the equation \( B(zx^{-1}) = Az - zx^{-1}Ax \). This operator has the essential properties used to obtain the representation of the previous theorem.

Reference

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