THE ARGUMENT OF AN ENTIRE FUNCTION

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THEOREM. Let $f(z)$ be an entire function of order $\rho$, and let $\phi(r)$ denote the number of points of the circle $|z| = r$ at which $f(z)$ is real. Then

$$\limsup_{r \to \infty} \frac{\log \phi(r)}{\log r} \geq \rho.$$  \hfill (1)

PROOF. The de la Vallée Poussin means of $f(z) = \sum b_n z^n$ are

$$V_n(z) = \frac{b_0}{2} + \sum_{n=1}^{\infty} \frac{C_{2n,n+p}/C_{2n,n}}{b_n} b_n z^n \quad (n = 0, 1, 2, \cdots).$$  \hfill (2)

It has been shown by Pólya and Schoenberg [1] that the curve $w = f(re^{it})$ crosses any straight line at least as often as the curve $w = V_n(re^{it})$. Taking this line to be the real axis, let $\phi(r), \phi_n(r)$ respectively, denote the number of points of $|z| = r$ at which $f(z), V_n(z)$ are real. Then

$$\phi(r) \geq \phi_n(r).$$

If $N_n(r)$ is the number of zeros of $V_n(z)$ in $|z| \leq r$, then by the argument principle, $\phi_n(r) \geq N_n(r)$ and thus

$$\phi(r) \geq N_n(r) \quad (n = 0, 1, 2, \cdots).$$

Suppose that in the circle $|z| \leq \rho_n$, $V_n(z)$ has at least $\rho$ zeros. Then for $r \geq \rho_n$

$$\phi(r) \geq \rho.$$  \hfill (3)

We have now the theorem of Montel (see [2, p. 113]) that in the circle

$$|z| \leq 1 + \max_{i \leq p} |a_i/a_n|^{1/(n-p+1)}$$

the polynomial

$$a_0 + a_1 z + \cdots + a_n z^n$$

has at least $\rho$ zeros. Applying this to (2), we can take

$$\rho_n \leq 1 + \max_{i \leq p} \left\{ C_{2n,n+i} \left| \frac{b_i}{b_n} \right| \right\}^{1/(n-p+1)}.$$  \hfill 488
Now if \( \epsilon > 0 \) is given, we have for all \( n \)
\[
|b_n| \leq An^{-n/(p+\epsilon)} \quad (A > 1)
\]
while, for infinitely many \( n \), \( |b_n| \geq n^{-n/(p-\epsilon)} \). Thus
\[
\rho_n \leq 1 + \max_{j \leq p} \left\{ A j^{-j/(p+\epsilon)} \right\} \frac{1}{n^{n/(p-\epsilon)}}
\]
for infinitely many \( n \). Hence for such \( n \),
\[
\rho_n \leq 1 + \left\{ A C_{2n,n+1}^{-1/(p-\epsilon)} \right\}^{1/(n-p)}
\]
and
\[
\leq 1 + \left\{ A^{1/n} C_{n+1}^{-1/(p-\epsilon)} \right\}^{n/(n-p)}.
\]
Now let \( \alpha \) be fixed, \( 0 < \alpha < 1 \), and take \( p = \alpha n \); then for infinitely many \( n \)
\[
\rho_n \leq 1 + \left\{ A^{1/n} A n^{-1/(p-\epsilon)} \right\}^{1/(1-\alpha)}
\]
and
\[
\leq \left\{ B n^{1/(p-\epsilon)} \right\}^{1/(1-\alpha)}.
\]
Hence from (3) with \( p = \alpha n \),
\[
\phi((B n^{1/(p-\epsilon)})^{1/(1-\alpha)}) \geq \alpha n
\]
and putting \( r = \left\{ B n^{1/(p-\epsilon)} \right\}^{1/(1-\alpha)} \), there is a sequence of values of \( r \) tending to infinity along which
\[
\phi(r) \geq \alpha B^{-(p-\epsilon)r(1-\alpha)(p-\epsilon)}
\]
whence
\[
\limsup_{r \to \infty} \frac{\log \phi(r)}{\log r} \geq (1 - \alpha)(\rho - \epsilon)
\]
and the result follows since \( \epsilon > 0 \) and \( 0 < \alpha < 1 \) were arbitrary.

We ask: (a) can the sign of inequality hold in (1)? (b) is it true that
\[
\limsup_{n \to \infty} \frac{\log n}{\log r_n} = \rho
\]
where \( r_n \) is the modulus of the zero of largest modulus of (2)?

REFERENCES


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