FLOWS ON SOME THREE DIMENSIONAL
HOMOGENEOUS SPACES

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1. Flows on surfaces of constant negative curvature have been investigated for some time. The geodesic flow [3] and the horocycle flow [2] have known minimal and ergodic properties. These flows may be looked at as flows induced on a three dimensional homogeneous space by a one parameter subgroup of a Lie group [4]. This idea has been carried further in [1; 5] where one parameter flows on general nilmanifolds are studied.

The manifolds considered here are all compact manifolds of the form $G/D$ where $G$ is a noncompact connected, simply connected three dimensional Lie group and $D$ a discrete uniform subgroup. If $\phi: T \to G$ is a one parameter subgroup of $G$, then the one parameter flow defined by $t(gD) = \phi(t)gD$, is an action of the reals on $G/D$. The classification as to which of these flows are minimal and which are ergodic is now complete. In this note we outline this classification; complete proofs will be presented elsewhere.

There are only three cases to consider: simple, nilpotent, and solvable but not nilpotent.

2. $G$ simple. If $G$ is simple and noncompact then its Lie algebra $\mathfrak{g}$ is isomorphic to the Lie algebra of the two by two real matrices with trace zero. Each one parameter subgroup of $G$ is of the form $\phi(t) = \exp \begin{bmatrix} tX \\ 0 \end{bmatrix}$, where $X \in \mathfrak{g}$. Let $G(2)$ be the group of all $2 \times 2$ real matrices of determinant one. $G$ is the universal covering group of $G(2)$ and we let $\eta$ be the covering homomorphism $\eta: G \to G(2)$.

**Theorem 1.** If $D$ is a discrete uniform subgroup of $G$ then the mapping $\psi: G/D \to G(2)/\eta(D)$ given by $\psi(gD) = \eta(g)\eta(D)$ is a finite covering and $\eta(D)$ is discrete.

**Theorem 2.** Let $G$ be the connected, simply connected, noncompact, three dimensional, simple Lie group; and let $D$ be a discrete uniform subgroup of $G$; and let $\phi(t) = \exp \begin{bmatrix} tX \\ 0 \end{bmatrix}$. The following statements hold:

1. If $X$ has real nonzero eigenvalues the one parameter flow induced

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by $\phi(t)$ on $G/D$ has infinitely many closed orbits, and is thus not minimal. This flow is ergodic.

(2) If $\bar{X}$ has only zero eigenvalues then the one parameter flow induced by $\phi(t)$ on $G/D$ is minimal and ergodic.

(3) If $\bar{X}$ has nonzero complex eigenvalues then the one parameter flow induced by $\phi(t)$ on $G/D$ is equivalent to an action on $G/D$ by the circle group and is thus neither minimal nor ergodic.

Theorem 1 is used to reduce the proof of Theorem 2 to the geodesic flow [3; 4] or the horocycle flow [2].

3. $G$ nilpotent. The complete classification of flows on nilmanifolds has been worked out in [1; 5] and, if $[G, G]$ is the commutator of $G$, it reads as follows:

**Theorem 3.** If $G$ is a connected, simply connected nilpotent Lie group; and $D$ a discrete uniform subgroup; and $\phi(t)$ a one parameter subgroup; then the flow induced by $\phi(t)$ on $G/D$ is always distal. Furthermore, it is ergodic (minimal) if and only if the flow induced on the torus $G/D[G, G]$ by $\phi(t)$ is ergodic (minimal).

4. $G$ solvable. We let $G_1$ be the set of matrices of the form

\[
\begin{pmatrix}
    \cos 2\pi z & \sin 2\pi z & 0 & x \\
    -\sin 2\pi z & \cos 2\pi z & 0 & y \\
    0 & 0 & 1 & z \\
    0 & 0 & 0 & 1
\end{pmatrix}
\]

where $x, y, z$ are real numbers. We let $G_2$ be the set of matrices of the form

\[
\begin{pmatrix}
    e^{kz} & 0 & 0 & x \\
    0 & e^{-kz} & 0 & y \\
    0 & 0 & 1 & z \\
    0 & 0 & 0 & 1
\end{pmatrix}
\]

where $x, y, z$ are real numbers, and $k$ is a fixed nonzero real number such that $e^{-k} + e^k$ is an integer.

**Theorem 4.** If $G$ is a connected, simply connected, three dimensional, solvable Lie group and $D$ is a discrete uniform subgroup, then one of the following is true.

(1) $G$ is nilpotent.

(2) $G$ is isomorphic to $G_1$ and $D$ is generated by
where \( n \) is a fixed integer, \( p \) is either 2, 3, 4, or 6, and
\[
\begin{pmatrix}
1 & 0 & 0 & u_1 \\
0 & 1 & 0 & u_2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\cos 2\pi n/p & \sin 2\pi n/p \\
-\sin 2\pi n/p & \cos 2\pi n/p \\
0 & 0 & 1 & n \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
u_1 \\
v_2 \\
1
\end{pmatrix}
\nequiv 0.

(3) \( G \) is isomorphic to \( G_1 \) and \( D \) is generated by
\[
\begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & u_1 \\
0 & 1 & 0 & u_2 \\
0 & 0 & 1 & n \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
where \( n \) is a fixed integer and \( u_1 \) and \( u_2 \) are fixed real numbers.

(4) \( G \) is isomorphic to \( G_2 \) and \( D \) is generated by
\[
\begin{pmatrix}
1 & 0 & 0 & u_1 \\
0 & 1 & 0 & u_2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & v_1 \\
0 & 1 & 0 & v_2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\cos^n & 0 & 0 & 0 \\
0 & \cos^{-n} & 0 & 0 \\
0 & 0 & 1 & n \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
where \( n \) is a fixed integer and
\[
\begin{pmatrix}
u_1 \\
v_2
\end{pmatrix}
\nequiv 0.

If, for the sake of brevity, we write the matrices of \( G_1 \) and \( G_2 \) as columns
\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
\]
then the one parameter subgroups of \( G_1 \) are in one of the following forms.

(1) \[
\begin{pmatrix}
at \\
b \\
0
\end{pmatrix}, \quad a \text{ and } b \text{ are real numbers.}
\]

(2) \[
\begin{pmatrix}
a \sin 2\pi ct + b[\cos 2\pi ct - 1] \\
b \sin 2\pi ct - a[\cos 2\pi ct - 1] \\
ct
\end{pmatrix},
\]
a, b, c, real numbers and \( c \neq 0 \). The one parameter subgroups of \( G_2 \) have the following forms.

\[
\begin{pmatrix}
a t \\ b t \\ 0
\end{pmatrix}, \quad a \text{ and } b \text{ real numbers.}
\]

\[
\begin{pmatrix}
a (e^{kt} - 1) \\ b (e^{-kt} - 1) \\ c t
\end{pmatrix}, \quad a, b \text{ and } c \text{ real numbers}
\]

and \( c \neq 0 \).

In either \( G_1 \) or \( G_2 \) we refer to these as one parameter groups of the first and second type respectively.

**Theorem 5.** If \( G \) is a connected, simply connected, three dimensional, non-nilpotent solvable Lie group, \( D \) a discrete uniform subgroup, and \( \phi: T \to G \) a one parameter subgroup, then one of the following is true.

1. If \( G \) is isomorphic to \( G_1 \), \( D \) as in Theorem 4 number (2), and \( \phi \) is of the first type, then the flow is neither ergodic nor minimal. If \( \phi \) is of the second type, then the flow is equivalent to the action of a circle group and is thus neither ergodic nor minimal.
2. If \( G \) is isomorphic to \( G_2 \), \( D \) as in Theorem 4 number (4), and \( \phi \) is of the first type, then the flow is neither ergodic nor minimal. If \( \phi \) is of the second type then the flow is ergodic and has a closed orbit and is thus not minimal.
3. If \( G \) is isomorphic to \( G_1 \), \( D \) as in Theorem 4 number (3), and \( \phi \) is of the first type, then the flow is neither ergodic nor minimal. If \( \phi \) is of the second type, then the flow is equivalent to a straight line flow on the three dimensional torus and thus has the same minimal and ergodic properties.

**Bibliography**


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