DERIVATIONS AND GENERATIONS OF FINITE EXTENSIONS

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Communicated by Nathan Jacobson, May 8, 1961

Let \( k \) be a given ground field, let \( \mathcal{F} \) denote the class of finite (= finitely generated) field extensions of \( k \) of tr.d. (= transcendence degree) \( \leq r \), and let \( n \) be the function defined on \( \mathcal{F} = \bigcup_0^\infty \mathcal{F}_r \) by: for any \( L \in \mathcal{F} \), \( n(L) = \) the minimal number of generators of \( L/k \). Classically it is known for suitable \( k \) that there exist purely transcendental extensions \( L/k \) having tr.d. 2, and containing impure subextensions of tr.d. 2, a fact which shows that in general \( n \) is not monotone in \( \mathcal{F} \) for all \( k \). The main result of this note establishes that these "counterexamples to Lüroth's theorem" constitute the only barriers to the monotonicity of \( n \) (see Theorem 2 for a precise statement). In particular it is demonstrated that \( n \) is montone on \( \mathcal{F}_1 \) for arbitrary \( k \), a result which appears new even when restricted to the subclass \( \mathcal{F}_0 \) of finite algebraic extensions of \( k \).

A result of independent (and possibly more general) interest, which is proved below, and which is essential to our proof of the statements above, is that \( \text{dim} \ \mathcal{D} \) is monotone on \( \mathcal{F} \), where for any \( L \in \mathcal{F} \), \( \mathcal{D}(L) \) is the vector space over \( L \) of \( k \)-derivations of \( L \). The connection between \( n \) and \( \text{dim} \ \mathcal{D} \) is given in the lemma.

**Lemma.** Let \( L/k \) be a finite extension of tr.d. \( r \), let \( s = \text{dim} \ \mathcal{D}(L) \), and let \( n = n(L) \). Then \( s \leq n \leq s+1 \); if \( s > r \), then \( n = s \).

**Proof.** It is known (e.g. [3, Theorem 41, p. 127]) that \( s \) is the smallest natural number\(^3\) such that there exist elements \( u_1, \ldots, u_s \in L \) such that \( L \) is separably algebraic over the field \( U = k(u_1, \ldots, u_s) \). Then \( L = U(a) \) for some \( a \in L \), so that \( s \leq n \leq s+1 \).

If \( s > r \), there exists \( u_q \) in the set \( S = \{ u_1, \ldots, u_s \} \) such that \( u_q \) is algebraically dependent over \( k \) on the complement of \( u_q \) in \( S \). For convenience renumber so that \( u_s \) is algebraic\(^4\) over the field \( T = k(u_1, \ldots, u_{s-1}) \). A short argument shows that \( L = U(a) \) for some

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\(^2\) Expressed in the other words: If \( L/k \) is not separably generated, then \( n(L) = \text{dim} \ \mathcal{D}(L) \).

\(^3\) Strictly speaking the notation should allow for the case \( s = 0 \). By agreement then \( U = k \).

\(^4\) In case \( s = 1 \) set \( T = k \).

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a ∈ L which is separably algebraic over \( T \). Thus \( L = T(u_\ast, a) \) is generated over \( T \) by two elements one of which is separable over \( T \). Then, by a classic result in field theory (cf. [2, §43, p. 138]), \( L = T(u_\ast') \) for suitable \( u_\ast' \in L \). Clearly then \( n = s \). Q.E.D.

If \( L/k \) is a finite extension, and \( L'/k \) a subextension, in general not every derivation of \( \mathcal{D}(L') \) can be extended to a derivation in \( \mathcal{D}(L) \). Nevertheless, the theorem below shows that \( \dim \mathcal{D} \) is a monotone function.

**Theorem 1.** Let \( L/k \) be a finite field extension, and let \( L'/k \) be any subextension. Then \( s = \dim \mathcal{D}(L) \geq s' = \dim \mathcal{D}(L') \).

**Proof.** Let \( r = \text{tr.d. } L/k \) and \( r' = \text{tr.d. } L'/k \). It is easy to see that it suffices to consider only the case \( r = r' \). For if \( r' < r \), then there exists a field extension \( L'' \) contained in \( L \) which is purely transcendental over \( L' \) and such that \( \text{tr.d. } L''/k = r \). Since \( L''/L' \) is a pure extension, every \( D' \in \mathcal{D}(L') \) is extendable to a derivation \( D'' \) in \( \mathcal{D}(L'') \).

It is an easy exercise to show that if \( D_1, \ldots, D'_t \) are linearly independent over \( L' \), then \( D_1', \ldots, D'_t \) are linearly independent over \( L'' \), so that \( \dim \mathcal{D}(L') \geq s' \). Hence it remains only to show that \( s \geq s' \) when \( r = r' \). An argument similar to the one just completed shows that \( s \geq s' \) when \( L/L' \) is separable. The proof now proceeds by induction on the algebraic degree \( |L:L'| \) of the extension \( L/L' \). One can therefore assume the theorem for all extensions \( L'' \) of \( k \) which contain \( L' \) and such that \( |L'' : L'| < |L : L'| \). Then clearly one can suppose that \( L' \) is a maximal proper subfield of \( L \). Since the separable case already has been decided, assume that \( k \) has characteristic \( p > 0 \), and that \( L/L' \) is inseparable. Then the maximality of \( L' \) shows that \( k(L^p) \subseteq L' \). By [1, p. 218] or [3, Theorem 41, p. 127], one has

\[
p^* = |L:k(L^p)|, \quad \text{and} \quad p'^* = |L':k(L'^p)|,
\]

so that the inclusions

\[
L \supset L' \supset k(L^p) \supset k(L'^p)
\]

together with the inequality

\[
|L:L'| \geq |k(L^p):k(L'^p) |
\]

yield the desired inequality \( p^* \geq p'^* \), that is, \( s \geq s' \). Q.E.D.

**Corollary.** Let \( L/k \) be a finite extension, and let \( L'/k \) be any subextension. Then, if either \( L/k \) or \( L'/k \) is not separably generated, then \( n(L) \geq n(L') \).
PROOF. Let $s = \dim \mathfrak{D}(L)$, $r = \text{tr.d. } L/k$, $n = n(L)$, and let $s', r'$, and $n'$ be the corresponding integers for $L'/k$. If $L'/k$ is not separably generated, neither is $L/k$, so that we can assume that $L/k$ is not separably generated in either case, that is, that $s \geq r + 1$. Then $n = s$ by the lemma, whence $n = s \geq s'$ by the theorem. If $n' = s'$, we are through, and if $n' \neq s'$, then $n' = s' + 1 = r' + 1$ by the lemma again. Since $r \geq r'$, this latter equality yields
\[ n = s \geq r + 1 \geq r' + 1 = s' + 1 = n', \]
as desired.

The corollary is surprising in view of the troublesomeness usually associated with nonseparably generated extensions.

Before stating the last result, it is convenient to make the definition: $k$ is an $(r)$-field in case no pure transcendental extension of $k$ of tr.d. $r$ contains an impure subextension of tr.d. $r$ over $k$. Clearly if $n$ is monotone in $\mathfrak{S}_r$, then $k$ must be an $(m)$-field, $m = 0, 1, \ldots, r$. Our next theorem establishes the converse.

**THEOREM 2.** If $k$ is an $(r)$-field, and if $L/k$ is a finite extension of tr.d. $r$, then $n = n(L) \geq n' = n(L')$ for any subextension $L'/k$ of $L/k$.

**PROOF.** Let $s, r, n$, and their primes be defined as in the corollary. If $s' > r'$, then $n > n'$ by the corollary. If $L'/k$ is purely transcendental, then trivially $n \geq n'$. Otherwise $s' = r'$ implies by the lemma that $n' = s' + 1 = r' + 1$. Then, since $k$ is an $(r)$-field, necessarily $n \geq r + 1 = r' + 1 = n'$, if $r = r'$. If $r > r'$, then clearly $n \geq r \geq r' + 1 = n'$. Q.E.D.

By definition, any field is a $(0)$-field, and, by Lüroth's theorem, any field is a $(1)$-field. Thus, the theorem implies the corollary:

**COROLLARY.** Let $k$ be an arbitrary field. Then $n$ is monotone in the class $\mathfrak{S}_1$ of finite extensions of tr.d. $\leq 1$ over $k$; in particular, $n$ is monotone in the class $\mathfrak{S}_0$ of finite algebraic extensions of $k$.

A consequence of Theorem 2 and the theorem of Castelnuovo-Zariski (see [4]) is the following:

**COROLLARY.** Let $k$ be an algebraically closed field of characteristic 0. Then $n$ is monotone in the class $\mathfrak{S}_2$ of finite extensions of tr.d. $\leq 2$ over $k$.

A possible value of Theorem 2 is that in order to show that a given field is not an $(r)$-field, it is possible to do this by showing that $n$ is not monotone on its finite extensions of tr.d. $r$, that is, one need not restrict one's attention to the pure transcendental extensions of $k$. 
REFERENCES


