A matrix $A$ is called conservative if $Ax$ is convergent (its limit is called $\lim_A x$) whenever $x$ is a convergent sequence, regular if $\lim_A x = \lim x$ for such $x$, coregular if conservative and $\chi(A) = \lim_{n \to \infty} \sum_{k=1}^n a_{nk}$, and conull if $\chi(A) = 0$. The terms coregular and conull were introduced in [2].

A regular matrix is coregular, as is any matrix equipotent with a regular one. However, there exist coregular matrices not equipotent with any regular matrix. The example, due to Zeller, is given in [3].

We present here an example of a quite different nature.

(An open problem in the field is that of characterizing FK spaces which have a right to be called coregular. That $\{1\}$ be separated from the linear closure of $\{\delta^*\}$ is necessary but not sufficient.)

Restricting ourselves, for convenience, to triangles ($a_{nn} \neq 0$, $a_{nk} = 0$ for $k > n$), let $c_A = \{x : Ax \text{ is convergent}\}$. Then $c_A$ is isomorphic with $c$, the space of convergent sequences, under $A : c_A \to c$. Thus $c_A$ becomes a Banach space. If $c_A = c_B = F$ say, the norms on $F$ due to $A, B$ are equivalent since $A = DB$ with $c_D = c$ and $\|x\|_4 = \|Ax\| \leq \|D\| \|Bx\| = \|D\| \|x\|_2$.

If the functional $\lim$ is continuous on $c \subset c_A$ we extend it by the Hahn-Banach theorem to be defined on all of $c_A$. By a construction of Mazur [1, Theorem 2, p. 45], we obtain a matrix $B$ with $\lim_B = \lim$ on $c$, and $c_B = c_A$. (See [2] for proof that $\lim$ satisfies Mazur's condition.)

Clearly $B$ is regular.

Conversely if such regular $B$ exists it follows that $\lim$ is continuous since $\lim = \lim_B$.

Thus, for our example, it is sufficient to construct a coregular matrix $A$ such that $\lim$ is not continuous on $c_A$.

Let $Y$ be the matrix such that $Yx = \{x_{n-1} + x_n\}$. Then $(1/2) Y$ is a regular triangle. Let $B$ be the matrix whose $n$th row is $\{t_1, t_2, \ldots, t_n, 0, 0, \ldots\}$ where $\{t_n\}$ is a suitably chosen sequence with $\sum |t_n| < \infty$. Then $B$ is in the radical of the Banach algebra $\Delta$ of conservative triangular matrices. (See [4].) Note that $Y$ has no inverse in this algebra. Finally, let $A = B + Y$. Then $A$ is coregular. The norm associated with $c_A$ is

$$\|x\| = \sup_n \left| \sum_{k=1}^{n-1} t_k x_k + x_{n-1} + x_n \right|.$$
We shall choose $t_n = (-1)^n/n^2$. Now to construct $x$ with $\lim |x|$ large and $\|x\|$ not large we shall, given any integer $m$, supply $x$ with $\lim x = -1/2 \log m$, $\|x\| < M$ where $M$ is an absolute constant which could easily be determined.

Namely, let

$$x_n = n(-1)^n - 1/2 \log n \quad \text{for } 1 \leq n \leq m,$$

$$= (2m - n)(-1)^n - 1/2 \log m \quad \text{for } m < n \leq 2m,$$

$$= -1/2 \log m \quad \text{for } n > m.$$

If the matrix $Y$ had been chosen so that $Yx = \{ sx_{n-1} + tx_n \}$ then for $t > s \geq 0$ we have that $Y^{-1} \in \Delta$ hence $A^{-1} \in \Delta$ since $B$ is in the radical. Thus $e_A = c$, lim is continuous and $\|x\| < 1$ implies $\lim x$ is less than some constant, depending on $s$, $t$ and $\{t_n\}$. This result also follows from Lemma 4.2 of [4]. There should be situations between these extremes in which $e_A \neq c$, yet a regular matrix coincident with $A$ exists. This may very well occur if all $t_n > 0$, and $s = t = 1$, but we are unable to say at present.

Added in proof. Research Problem 3 (Bull. Amer. Math. Soc. vol. 67 (1961) p. 355) by Albert Wilansky, which inspired this article, has also been solved by Lawrence Shepp.

References


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