COMBINATORIAL EMBEDDINGS OF MANIFOLDS

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The following results on embedding manifolds resemble in their form Dehn’s Lemma, the Sphere Theorem, and, especially, embedding theorems obtained for differentiable manifolds by A. Haefliger [1].

Let $M, Q$ be finite combinatorial manifolds of dimensions $m$ and $q$, respectively. Let $\partial M, \partial Q$ be their boundaries (possibly empty), and let $f: M \to Q$ be a piecewise linear map. We define $\text{sing}(f)$ to be the closure in $M$ of the set $\{x \in M; f^{-1}(x) \neq x\}$. Let $R = M \cap f^{-1}(\partial Q)$, $S$ be a regular neighbourhood of $R$ in $M$ (see [3]) and $T = M - S$.

**THEOREM 1.** Of the following conditions, (i), (ii), (iii), and any one of (iv), (v), (vi) are sufficient to ensure the existence of a piecewise linear embedding $g: M \subset Q$ such that $g$ is homotopic to $f$ rel. $\partial M$:

(i) $q \geq m + 3$,
(ii) $M$ is $(2m - q)$ connected,
(iii) $Q$ is $(2m - q + 1)$ connected,
(iv) $f(M) \subset \partial Q$,
(v) $\text{sing}(f) \cap M = \emptyset$ and $T$ is $(3m - 2q + 1)$ connected,
(vi) $\text{sing}(f) \cap R = \emptyset$ and $T$ is $(2m - q - 1)$ connected.

**REMARKS.** If $M = \emptyset$, we regard condition (iv) as being trivially satisfied. If $f(M) \subset \partial Q$, we have the convention that the only regular neighbourhood of the empty set is the empty set, and so $T = M$.

In particular:

**COROLLARY 2.** Any element of $\Pi_m(Q)$, where $Q$ is $(2m - q + 1)$ connected ($q \geq m + 3$), may be represented by a piecewise linear embedding of $S^m$.

**THEOREM 3.** If $q = 2m$, there exists a piecewise linear embedding

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g: $M \subset Q$ such that $g$ is homotopic to $f$ rel. $M$ provided that $M$ is connected, $\hat{M}$ is non empty and $f(\hat{M})$ is not contained in $\hat{Q}$.

We recall that if $X, X_0$ are two complexes, $X$ is said to contract to $X_0$ by an elementary contraction if $X = X_0 + \sigma + \tau$ where $\sigma$ and $\tau$ are simplexes (boundaries $\partial \sigma$ and $\partial \tau$ respectively), $\sigma = a \partial \sigma$ (a a vertex), $a \partial \sigma \in X_0$, and $\tau \in X_0$. We say that, if $X, X_0$ are subcomplexes of a combinatorial manifold $A$, $X$ contracts to $X_0$ by an admissible elementary contraction in $A$ if $X$ contracts to $X_0$ by an elementary contraction $X = X_0 + \sigma + \tau$, and the following case does not hold: $\tau \in \partial A$, $\sigma \in \partial A$ ($\partial A$ is the boundary of $A$). We say that $X$ is admissibly collapsible in $A$ if $X$ contracts to a point by a series of admissible elementary contractions in $A$.

The proof of Theorem 1 leans heavily on the following Lemma, which is an extension of a result of E. C. Zeeman [5; 6] to cover the case of a bounded manifold.

**Lemma.** If $A$ is a $t$-dimensional finite combinatorial manifold with boundary $\partial A$, $X$ an admissibly collapsible subcomplex of $A$, and $K$ an $s$-dimensional subcomplex of $A$ such that $K \cap \partial A = J$, and if further

1. $J$ is $r$-dimensional, $r < s$,
2. $t \geq s + 3$,
3. $A$ is $s$-connected and $\partial A$ $r$-connected,

then, in a suitable subdivision $\beta A$ of $A$, there exists an $(s+1)$-dimensional subcomplex $L$ of $\beta A$ such that $\beta K \subset L$, and $\beta X \cup L$ is admissibly collapsible in $\beta A$.

Theorem 3 is proved by an argument practically identical to that of Lemma 2.7 in [2].

There are counterexamples to show the necessity for some, perhaps weaker, conditions like (ii) and (iii) of Theorem 1.

**Counterexample 1.** (Condition (ii) for $M$ closed.) $M$ = real projective space of dimension $2r$, $r > 0$, $Q$ = a combinatorial $q$-ball $B^q$, where $q < 2 \cdot 2r$, $f$ = the map taking $M$ to a point of $Q$. It is well known that there is no embedding of $M$ in $Q$.

**Counterexample 2.** (Condition (ii) for $M$ bounded.) $M = S^r \times B^{s+1}$, $\hat{M} = S^r \times S^s$, $Q = B^{2s+2}$, $\hat{Q} = S^{2s+1}$, where $r$ and $s$ are such that $S^r$ admits a continuous $r$-field of tangent vectors, $f$ is a piecewise linear map of $M$ in $Q$, taking $\hat{M}$ into $\hat{Q}$ in such a way that two cross-sectional copies of $S^r$ become homologically once linked in $S^{2s+1}$ (see [4; 6]). There exists no embedding $g$ of $M$ in $Q$ with $g|_M = f|_M$ for $M$.

**Sublemma.** If $S^r_1, S^r_2 \subset S^{2s+1}$ are homologically once linked combi-
torial $s$-spheres, there do not exist in $B^{2s+2}$ combinatorial $(s+1)$-balls $B^{s+1}_1, B^{s+1}_2$ spanning $S^1, S^2$ and such that $B^{s+1}_1 \cap B^{s+1}_2 = \emptyset$.

Counterexample 3. (Condition (iii)). There exists a $2m$-dimensional orientable finite combinatorial manifold $Q(m \geq 2)$ such that $\Pi_m(Q)$ is nonzero, and no nonzero element of it can be represented by a piecewise linear embedding of $S^m$ in $Q$. D. B. A. Epstein suggested to me some time ago as a candidate for a counterexample to the Sphere Theorem in four dimensions an $S^2$ in $E^4$, which cuts itself orthogonally in just one point, fattened in $E^4$. This is essentially the manifold $Q$ for $m = 2$. The counterexample is proved by studying the universal covering space of $Q$.

Counterexample 4. $M = B^m$, $\hat{M} = S^{m-1}$, $Q = S^1 \times B^{2m-1}$, $\hat{Q} = S^1 \times S^{2m-2}$, $f: M \rightarrow Q$ is a piecewise linear map such that $f(\hat{M}) \subset \hat{Q}$ links itself once homologically around the $S^1$ of $\hat{Q}$. There exists no piecewise linear embedding $g$ of $M$ in $Q$ such that $f|\hat{M} = g|\hat{M}$. This follows from Counterexample 3, or may be proved independently by applying the Sublemma to the universal covering space of $Q$. It shows the necessity for the condition that $f(\hat{M})$ should not be contained in $\hat{Q}$ in Theorem 3.

References


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