ENTIRE FUNCTIONS AND INTEGRAL TRANSFORMS

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If \( E(z) \) is an entire function which satisfies

\[
|E(z)| < |E(x)|
\]

for \( y > 0 \ (z = x + iy) \), let \( \mathcal{H}(E) \) be the corresponding Hilbert space of entire functions \( F(z) \) such that

\[
\|F\|_2^2 = \int |F(t)/E(i)|^2 \, dt < \infty
\]

and

\[
|F(z)|^2 \leq \|F\|^2 \left[ |E(z)|^2 - |E(z)|^2 \right]/[2\pi(\bar{z} - z)]
\]

for all complex \( z \). The space is introduced in [7], where it is characterized by three axioms. If \( E(a, z) \) and \( E(b, z) \) are entire functions which satisfy (1), then \( \mathcal{H}(E(a)) \) will be contained isometrically in \( \mathcal{H}(E(b)) \) if these functions satisfy the hypotheses of Theorem VII of [8]. Isometric inclusions of spaces of entire functions are a basic idea in [9] and [10]. A fundamental property of these inclusions has only now become available.

**Theorem I.** If \( E(a, z), E(b, z), \) and \( E(c, z) \) are entire functions which satisfy (1) and have no real zeros, and if \( \mathcal{H}(E(a)) \) and \( \mathcal{H}(E(b)) \) are contained isometrically in \( \mathcal{H}(E(c)) \), then either \( \mathcal{H}(E(a)) \) contains \( \mathcal{H}(E(b)) \) or \( \mathcal{H}(E(b)) \) contains \( \mathcal{H}(E(a)) \).

The formal proof depends on techniques of [2] and [3] for handling difference quotients. To make it precise, one must show that if \( f(z) \) and \( g(z) \) are entire functions of minimal exponential type such that

\[
|yf(z)g(z)| \leq |f(z)| + |g(z)|
\]

for all complex \( z \), then \( f(z)g(z) \) vanishes identically. This is proved by a method of Carleman, for whose explanation we are indebted to M. Heins [16]. By Theorem III of [10], the theorem has applications for certain kinds of integral transforms.

**Theorem II.** Let \( u(x) \) and \( v(x) \) be square integrable functions defined in \([0, 1]\), such that

\[
u(x)v(x) = \bar{v}(x)u(x)
\]
a.e., and which are essentially linearly independent when restricted to
any subinterval of $[0, 1]$. Let $T$ be the bounded linear transformation of $L^2(0, 1)$ into itself defined by $T$: $g \rightarrow f$ if

$$f(x) = \int_x^1 g(t) [u(x)v(t) - v(x)u(t)] dt$$

for almost all values of $x$. Let $M$ be a closed subspace of $L^2(0, 1)$ which is invariant under $T$ in the sense that $Tg$ belongs to $M$ whenever $g$ belongs to $M$. Then, $M$ is characterized by a number $a$ in $[0, 1]$ and coincides with the set of functions which vanish a.e. for $x \geq a$.

The same conclusion is available from the work of Kalisch [17] when $u(x)$ and $v(x)$ satisfy additional differentiability conditions. The point of Theorem II is that no such restrictions are necessary. Theorem II may be used to give a proof of uniqueness in the inverse Sturm-Liouville problem studied by Levinson [19].

**Theorem III.** Let $\psi(x)$ be a uniformly continuous, increasing function of real $x$ such that

$$\int (1 + t^2)^{-1} |\psi(t) - \tau t|^2 dt < \infty$$

for some number $\tau > 0$. If $0 < a < \tau$, then there is a measure $\mu$ of finite total variation, supported in the points $t$ where $\psi(t) \equiv 0$ modulo $\pi$, such that $\int e^{it\alpha} d\mu(t)$ vanishes in $[-a, a]$ and does not vanish identically. Furthermore, the measure may be chosen of this special form: There is an entire function $S(z)$ of exponential type $a$ which is real for real $z$ and has only real simple zeros, all at points $t$ where $\psi(t) \equiv 0$ modulo $\pi$, and

(2) $$\int (1 + t^2)^{-1} \log^+ |S(t)| dt < \infty$$

and

$$\sum_{S(t) = 0} |S'(t)|^{-1} < \infty.$$ 

The measure $\mu$ is supported in the zeros of $S(z)$ and has mass $S'(t)^{-1}$ at each such zero $t$.

The formal part of the proof depends on the formula of [6] to obtain a measure, and on the convexity methods of [4] and [5] to obtain an entire function. To implement these procedures, we use a theorem of Beurling and Malliavin [20]: If $K(z)$ is an entire function of exponential type which satisfies (2), then for each $a > 0$ there is a nonzero entire function $F(z)$ of exponential type $a$, bounded on the
real axis, such that $K(z)F(z)$ is bounded on the real axis. Under the hypotheses of Theorem III, an entire function of minimal exponential type, which remains bounded on the set of points $t$ where $\psi(t)=0$ modulo $\pi$, is necessarily a constant. We should like to acknowledge our indebtedness to Chapter VIII of Levinson [18], which suggested the above theorem. The results of Levinson, Chapter IX, can be significantly bettered on using another theorem of Levinson, as it is formulated in [3]. The trick is to use Theorem XII of [9] to convert a result on nonvanishing Fourier transforms into an existence theorem for entire functions of minimal exponential type.

**Theorem IV.** Let $(a_n, b_n)$ be a sequence of disjoint intervals to the right of $x=1$ with lengths $b_n-a_n$ bounded away from zero and with

$$\sum (b_n - a_n)^2a_n^{-1}b_n^{-1} = \infty.$$  

Then there exists an entire function of minimal exponential type which remains bounded on the real complement of $\bigcup(a_n, b_n)$ and is not a constant.

**References**


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