ON THE REPRESENTATION PROBLEM FOR STATIONARY
STOCHASTIC PROCESSES WITH TRIVIAL TAIL FIELD

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Let \{X_n\} be a real valued strictly stationary stochastic process on
the probability space \((\Omega, \Sigma, P)\) and let \{\xi_n\} be an independent se­
quence of random variables uniformly distributed on \([0, 1]\) where
\(n = 0, \pm 1, \cdots\). When does there exist a function \(f\) on the sequence
\{\xi_n\} such that the sequences \{X_n\} and \{f(T^n\xi)\} have the same
probability structure where \(\xi = (\cdots, \xi_{-1}, \xi_0, \xi_1, \cdots)\) and \(T\xi\)
= (\cdots, \xi_0, \xi_1, \xi_2, \cdots) \text{ (i.e. such that the joint distribution of}
\(X_{i_1}, \cdots, X_{i_k}\) is the same as the joint distribution of \(f(T^n\xi), \cdots, \)
f\(T^n\xi\) for all \(k\) and all sequences \(i_1, \cdots, i_k)\)?

Let \(\Sigma_n\) be the smallest \(\sigma\)-field of subsets of \(\Omega\) with respect to which
\(X_k\) is measurable for all \(k \leq n\) and let \(\Sigma_\infty = \cap \Sigma_n\). \(\Sigma_\infty\) is called the tail
field of the process \{\(X_n\)\} and is said to be trivial if \(A \in \Sigma_\infty\) implies
\(P(A) = 0\) or 1. It has been shown (see [1] and [2]) that if \{\(X_n\)\} is a
stationary Markov chain with a denumerable state space and whose
tail field is trivial then a representation of the above type holds and
in fact \(f(T^n\xi) = f(\cdots, \xi_{n-1}, \xi_n)\).²

By use of a fairly simple transformation an arbitrary stationary
process \{\(X_n\)\} with trivial tail field can be converted to a stationary
Markov process \{\(Y_n\)\} with trivial tail field and from which the \{\(X_n\)\}
process can be recovered. Thus the seeming preoccupation with
Markov processes.

The following theorem generalizes Rosenblatt's results to a class of
Markov process with nondenumerable state space. \(\overline{P}\) is the stationary
measure induced by the process on the state space and \(P_X(A')\) is
the stationary conditional probability that \(X_n \in A'\) given \(X_{n-1} = X\).

**Theorem.** Let \{\(X_n\), \(n = 0, \pm 1, \cdots\) be a real stationary Markov
process such that
(i) \(\Sigma_\infty\) is trivial.
(ii) There exist Borel subsets \(A\) and \(B\) of the state space and a non­
negative measure \(\phi\) on the state space such that \(\overline{P}(B) > 0, \phi(A) > 0,\) and
for all \(X \in B\) and \(A' \subset A\) we have \(P_X(A') \geq \phi(A')\).

¹ This work was performed under the auspices of the United States Atomic Energy
Commission.
² A stationary Markov chain with denumerable state space has a trivial tail field
if and only if it is ergodic and aperiodic.
Then if \( \{ \xi_n \} \) is an independent sequence of random variables uniformly distributed \([0, 1]\) there exists a function \( g = g(\cdots, \xi_{-1}, \xi_0) \) such that the sequences \( \{ X_n \} \) and \( \{ g(\cdots, \xi_{n-1}, \xi_n) \} \) have the same probability structure.

**Corollary 1.** In the above theorem it is sufficient to replace condition (ii) with

(iiia) The state space of \( \{ X_n \} \) has an atom under the stationary probability \( \bar{P} \).

**Corollary 2.** If \( \{ X_n \} \) is a stationary ergodic aperiodic Markov chain with a denumerable state space then conditions (i) and (ii) hold and the above theorem is true.

Detailed proofs will appear elsewhere.

**References**


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