ON VANISHING ALGEBRAS

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Let $G$ be a locally compact group with left invariant Haar measure $m$. For any measurable subset $S$ of $G$, define $L_S$ to be that subset of $L^1(G)$ consisting of all functions which vanish (a.e.) on the complement of $S$. When $L_S$ forms an algebra, we call it a vanishing algebra. It is known that when $S$ is a semigroup l.a.e. (i.e., there exists a semigroup $T$ in $G$ such that $S = T$ locally almost everywhere), $L_S$ is a vanishing algebra. The following theorem gives an answer to a problem formulated by A. Simon [2]:

**Theorem 1.** Suppose $G$ is unimodular. If $L_S$ is a vanishing algebra and $S$ is contained in a σ-compact subset of $G$, then $S$ is a semigroup a.e.

**Corollary 1.** Suppose $G$ is compact. Then, if $L_S$ is a vanishing algebra, $S$ is a semigroup a.e.

**Corollary 2.** Suppose $G$ is abelian and generated by some compact neighborhood of the identity element of $G$. Then, if $L_S$ is a vanishing algebra, $S$ is a semigroup a.e.

The proof of Theorem 1 also gives the following more general and involved statement:

**Theorem 2.** Let $L_S$ be a vanishing algebra. Suppose there exists a directed set $\{ U_i, i \in I \}$ of symmetric neighborhoods of the identity element $e$ with finite measures, having the property that for almost all the points $x$ of $S$ there exists a $j_x \in I$ such that $m(S \cap x U_j)$ and $m(x^{-1} U_i \cap S^{-1})$ are both $> m(U_i)/2$ as $i \geq j_x$. Then $S$ is a semigroup l.a.e. If, in addition, $S$ is contained in a σ-compact subset of $G$, then $S$ is a semigroup a.e.

**Theorem 3.** If $L_S$ is a self-adjoint vanishing algebra, then $S$ is a group l.a.e. If, in addition, $S$ is contained in a σ-compact subset of $G$, then $S$ is a group a.e.

**Theorem 4.** Let $L_S$ be a vanishing algebra. If $S$ is open, then $S$ is a semigroup l.a.e. If, in addition, $S$ is contained in a σ-compact subset of $G$, then $S$ is a semigroup a.e.

**Theorem 5.** If $L_S$ is a maximal vanishing algebra, then $S$ is a closed

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semigroup l.a.e. If, in addition, $S$ is contained in a $\sigma$-compact subset of $G$, then $S$ is a closed semigroup a.e.

**Corollary 3.** Let $G$ be abelian and generated by some compact neighborhood of the identity element of $G$. If there exists a vanishing algebra $L_S$ which is a maximal subalgebra in $L^1(G)$, then $G$ is either the additive group of real numbers or the discrete integer group.

**References**


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