SUPPORTS OF A CONVEX FUNCTION

BY E. EISENBERG

Communicated by J. V. Wehausen, January 20, 1962

Let $C$ be a real, symmetric, $m \times m$, positive-semi-definite matrix. Let $R^m = \{(x_1, \ldots, x_m) | x_i$ is a real number, $i = 1, \ldots, m\}$, and let $K \subseteq R^m$ be a polyhedral convex cone, i.e., there exists a real $m \times n$ matrix $A$ such that $K = \{x | x \in R^m$ and $xA \leq 0\}$. Consider the function $\psi: K \rightarrow R$ defined by $\psi(x) = (xCx^T)^{1/2}$ for all $x \in K$. We wish to characterize the set, $U$, of all supports of $\psi$, where

\[ U = R^m \cap \{ u | x \in K \Rightarrow u x^T \leq (xCx^T)^{1/2} \}. \]

Let $R^n_+ = R^n \cap \{ \pi | \pi \geq 0 \}$ and consider the set

\[ V = \{ v | \exists x \in R^m, \pi \in R^n_+ \text{ and } v = \pi A^T + xC, xC x^T \leq 1, \ xA \leq 0 \}. \]

We shall demonstrate:

**Theorem.** $U = V$.

We first show:

**Lemma 1.** $x, y \in R^m \Rightarrow (xCy^T)^2 \leq (xCx^T)(yCy^T)$.

**Proof.** If $x, y \in R^m$ consider the polynomial $\rho(\lambda) = \lambda^2 xCx^T + 2\lambda xCy^T + yCy^T = (x+\lambda y)^T C (x+\lambda y)$. Since $C$ is positive-semi-definite, $\rho(\lambda) \geq 0$ for all real numbers $\lambda$, and thus the discriminant of $\rho$ is nonpositive, i.e.,

\[ 4(xCy^T)^2 - 4(xCx^T)(yCy^T) \leq 0. \]

q.e.d.

As an immediate application of Lemma 1 we show:

**Lemma 2.** $V \subseteq U$.

---

\[ ^1 \text{This research was supported in part by the Office of Naval Research under contract Nonr-222(83) with the University of California. Reproduction in whole or in part, is permitted for any purpose of the United States Government.} \]
PROOF. Let \( v \in V \), then there exist \( x \in \mathbb{R}^n, \pi \in \mathbb{R}^n_+ \) such that \( v = \pi A^T + xC, \ xCx^T \leq 1 \). Now if \( y \in \mathbb{R}^n, yA \leq 0 \), then \( vy^T = yA^T + xCy^T \) and \( vy^T \leq xCy^T \), because \( yA \leq 0, \ \pi^T \geq 0 \) and \( yA\pi^T \leq 0 \). Thus, \( vy^T \leq (xCx^T)^{1/2}(yCy^T)^{1/2} \), by Lemma 1, and \( vy^T \leq (yCy^T)^{1/2} \), because \( xCx^T \leq 1 \). Thus, \( v \in U \). q.e.d.

From the fact that \( C \) is positive-semi-definite, it follows that:

**Lemma 3.** The set \( V \) is convex.

**Proof.** If \( x_k \in \mathbb{R}^n, \pi_k \in \mathbb{R}^n_+, \ x_kA \leq 0, \ \pi_k = \pi_k A^T + x_k C, \ x_k C x_k^T \leq 1 \), \( \lambda_k \in \mathbb{R}_+ \) for \( k = 1, 2 \) and \( \lambda_1 + \lambda_2 = 1 \), then: \( \lambda_1 u_1 + \lambda_2 u_2 = (\lambda_1 \pi_1 + \lambda_2 \pi_2) A^T + (\lambda_1 x_1 + \lambda_2 x_2) C \), \( (\lambda_1 x_1 + \lambda_2 x_2) \leq 0 \), \( \lambda_1 x_1 + \lambda_2 x_2 \in \mathbb{R}^n, \ \lambda_1 \pi_1 + \lambda_2 \pi_2 \in \mathbb{R}^n_+ \), and

\[
(\lambda_1 x_1 + \lambda_2 x_2) C (\lambda_1 x_1 + \lambda_2 x_2) - 1
\]

\[
\leq (\lambda_1 x_1 + \lambda_2 x_2) C (\lambda_1 x_1 + \lambda_2 x_2)^T - \lambda_1 x_1 C x_1^T - \lambda_2 x_2 C x_2^T
\]

\[
= -\lambda_1 \lambda_2 [x_1 C x_1^T - 2 x_1 C x_2^T + x_2 C x_2^T]
\]

\[
= -\lambda_1 \lambda_2 (x_1 - x_2) C (x_1 - x_2)^T \leq 0,
\]

because \( C \) is positive-semi-definite. q.e.d.

**Lemma 4.** The set \( V \) is closed.

**Proof.** Let \( \{w_k\} \) be a sequence with \( w_k \in \mathbb{R}^n, k = 1, 2, \ldots \). We define the (pseudo) norm of \( w_k \), denoted \( \|w_k\| \), to be the smallest non-negative integer \( p \) such that there exists a \( k_0 \) and for all \( k \geq k_0 \), \( x_k \) has at most \( p \) nonzero components. Now, suppose \( u \) is in the closure of \( V \), i.e., there exist sequences \( \{u_k\}, \{\pi_k\} \) and \( \{x_k\} \) such that

\[
\pi_k \in \mathbb{R}^n_+, \ x_k \in \mathbb{R}^n, \ u_k = \pi_k A^T + x_k C,
\]

\[
x_k A \leq 0 \quad \text{and} \quad y_k C x_k^T \leq 1, \quad k = 1, 2, \ldots
\]

and \( \{u_k\} \) converges to \( u \).

Suppose the sequence \( \{x_k\} \) is bounded, then we may assume, having taken an appropriate subsequence, that for some \( x \in \mathbb{R}^n, \{x_k\} \rightarrow x \) and thus, by (3), \( xA \leq 0 \) and \( xCx^T \leq 1 \). Now, \( yA \leq 0 \Rightarrow u_k y^T - x_k C y^T = \pi_k A^T y^T = yA \pi_k^T \leq 0 \), all \( k \Rightarrow u y^T - x C y^T \leq 0 \). Thus the system,

\[
y \in \mathbb{R}^n
\]

\[
yA \leq 0
\]

\[
(u - xC)y^T > 0
\]

has no solution and by the usual feasibility theorem for linear inequalities (see e.g. [4] or [5]) the system:
$\pi \in \mathbb{R}^n_+$

$\pi A^T = u - xC$

has a solution, and thus $u \in V$.

We have just demonstrated that if \( \{x_k\} \) is bounded, then $u \in V$. Since $|\{x_k\}| + |\{x_kA\}| \leq m + n$, it is always possible to choose \( \{x_k\} \) and \( \{\pi_k\} \) satisfying (3) and such that $|\{x_k\}| + |\{x_kA\}|$ is minimal. We shall show next that if \( \{x_k\} \), \( \{\pi_k\} \) are so chosen, then \( x_k \) is indeed bounded, thus completing the proof. Suppose then that \( \{x_k\} \) is not bounded, i.e., \( \exists \) a subsequence such that $|x_k| = (x_kx_k^T)^{1/2} \to \infty$, and we may assume that $|x_k| > 0$ for all $k$. Let

$$z_k = \frac{x_k}{|x_k|}, \quad k = 1, 2, \ldots,$$

then \( \{z_k\} \) is bounded and we may assume that there is a $z \in \mathbb{R}^n$ such that the $z_k$ converge to $z$ and $|z| = 1$. From (3) it follows that $z_kA \leq 0$ and $z_kCz_k^T \leq 1/|x_k|$ for all $k$. Thus, $zA \leq 0$ and $zCz^T \leq 0$. But then, from Lemma 1, $zCz^T = 0$ for all $y \in \mathbb{R}^m$, and $zC = 0$. Summarizing:

\begin{equation}
(4) \quad z \in \mathbb{R}^m, \quad zA \leq 0, \quad zC = 0.
\end{equation}

Note that if $z$ has a nonzero component, then infinitely many $x_k$'s must have the same component nonzero, this follows from the fact that $z$ is the limit of $x_k/|x_k|$. As a consequence, if \( \{\lambda_k\} \) is any sequence of real numbers, then $|\{x_k + \lambda_kz\}| \leq |\{x_k\}|$. If $zA \neq 0$, and $a'$, $j = 1, \ldots, n$, denotes the $j$th column of $A$, let

$$\lambda_k = \max \left\{ \frac{z_k^2a'^j}{z_k^2} : j = 1, \ldots, n \text{ and } za'^j < 0 \right\}.$$

Then we may replace, in (3), $x_k$ by $x_k + \lambda_kz$ because $\lambda_kz^2a'^j + x_ka'^j \leq 0$ for all $j$, and $(x_k + \lambda_kz)A \leq 0$, also $zC = 0$ and thus $(x_k + \lambda_kz)C = x_kC$, $(x_k + \lambda_kz)(x_k + \lambda_kz)^T = x_kCx_k^T \leq 1$. However each $(x_k + \lambda_kz)A$ has at least one more zero component than $x_kA$, contradicting the minimality of $|\{x_k\}| + |\{x_kA\}|$. Thus, $zA = 0$ and we may replace, in (3), $x_k$ by $x_k + \lambda_kz$ for an arbitrary sequence \( \{\lambda_k\} \). But $z \neq 0$ and we can define $\lambda_k$ so that $x_k + \lambda_kz$ has at least one more zero component than $x_k$ has, thus $|\{x_k + \lambda_kz\}| < |\{x_k\}|$. However, $(x_k + \lambda_kz)A = x_kA$, and $|\{(x_k + \lambda_kz)A\}| = |\{x_kA\}|$, contradicting the minimality assumption. q.e.d.

Lastly, we show:

**Lemma 5.** $U \subset V$. 
Proof. Suppose \( u \in V \). By Lemmas 3 and 4 \( V \) is a closed convex set, hence there is a hyperplane which separates \( u \) strongly from \( V \) (see [4]). Thus there exist \( x \in \mathbb{R}^m \) and \( \alpha \in \mathbb{R} \) such that

\[
ux^T > \alpha \geq vx^T,
\]

all \( v \in V \).

Now, if \( \pi \in \mathbb{R}^n_+ \) then \( v = \pi A^T \) is in \( V \) (taking \( x = 0 \) in the definition of \( V \)). Thus \( xA\pi^T = \pi A^Tx^T \leq \alpha \) for all \( \pi \in \mathbb{R}^n_+ \), and \( xA \leq 0, \; x \in K \). Also \( v = 0 \) is in \( V \), so that \( \alpha \geq 0 \). If \( u \in U \) then \( 0 \leq \alpha < ux^T \leq (xCx^T)^{1/2} \), thus \( xCx^T > 0 \) and

\[
v = \frac{xC}{(xCx^T)^{1/2}} \in V,
\]

consequently,

\[
(xCx^T)^{1/2} > \alpha \geq \frac{xCx^T}{(xCx^T)^{1/2}} = (xCx^T)^{1/2}
\]

a contradiction. Thus \( u \in U \). q.e.d.

Note. A direct application of Lemmas 2 and 5 yields the theorem stated at the beginning.

References


University of California, Berkeley