SOME TWO-GENERATOR ONE-RELATOR NON-HOPFIAN GROUPS

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In 1951 Graham Higman claimed (in [1]) that every finitely generated group with a single defining relation is Hopfian,\(^2\) attributing this fact to B. H. Neumann and Hanna Neumann. However we shall show that this is not, in any way, the case. For example the group

\[(1) \quad G = \langle a, b; a^{-1}b^2a = b^3 \rangle \]

is non-Hopfian. Hence the following question of B. H. Neumann [2, p. 545] has a negative answer: Is every two-generator non-Hopfian group infinitely related?

This group \(G\) turns out to be useful for deciding a somewhat different kind of question. For Graham Higman\(^3\) has pointed out that \(G\) can, of course, be generated by \(a\) and \(b\). However it transpires that in terms of these generators \(G\) requires more than one relation to define it. Thus Higman has produced a counter-example to the following well-known conjecture: Let \(G\) be generated by \(n\) elements \(a_1, a_2, \ldots, a_n\) and let \(r\) be the least number in any set of defining relations between \(a_1, a_2, \ldots, a_n\). Then \(n - r\) is an invariant of \(G\) (i.e. does not depend on the particular basis \(a_1, a_2, \ldots, a_n\)). This conjecture has received some attention in the past; indeed there is a "proof" of it by Petresco [3].

The group defined by (1) is clearly only one of a larger family of groups of the kind

\[(2) \quad G = \langle a, b; a^{-1}b^2a = b^m \rangle. \]

It is convenient at this point to introduce a definition. Thus we say two nonzero integers \(l\) and \(m\) are meshed if either

(i) \(l\) or \(m\) divides the other,

or,

(ii) \(l\) and \(m\) have precisely the same prime divisors. This definition enables us to distinguish easily between the Hopfian and the non-Hopfian groups in the family of groups (2). For the following theorem holds.

**Theorem 1.** Let \(l\) and \(m\) be nonzero integers. Then

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\(^2\) A group \(G\) is Hopfian if \(G/N \cong G\) implies \(N=1\); otherwise \(G\) is non-Hopfian.

\(^3\) In a letter.
\[ G = \langle a, b; a^{-1}b^l a = b^m \rangle \]

is Hopfian if and only if \( l \) and \( m \) are meshed.

The proof of Theorem 1 is in three parts. Thus we prove

(a) if \( l \) or \( m \) divides the other, then \( G \) is residually finite\(^4\) and therefore Hopfian (Mal'cev [4]);

(b) if \( l \) and \( m \) are meshed but neither divides the other, then every endomorphism of \( G \) is an automorphism and so \( G \) is Hopfian;

(c) if \( l \) and \( m \) are not meshed, then \( G \) is non-Hopfian.

It is perhaps worthwhile to sketch the proof of (c). Here we may assume, without loss of generality, the existence of a prime \( p \) dividing \( l \) but not \( m \). Hence the mapping

\[ \eta: a \rightarrow a, \quad b \rightarrow b^p \]

defines an endomorphism of \( G \). Now it follows from the work of Magnus [5; 6] that

\[ [b^{l/p}, a]^{p^{l-m}} \neq 1. \]

However

\[ ([b^{l/p}, a]^{p^{l-m}}) \eta = [b^l, a]^{p^{l-m}} = 1. \]

Therefore the kernel \( K \) of \( \eta \) is nontrivial and as

\[ G(=G_\eta) \cong G/K \]

we have proved \( G \) is non-Hopfian.

The following theorem is a direct consequence of Theorem 1. It illustrates strikingly that hopficity is a finiteness condition of the weakest kind.

**Theorem 2.** The group

\[ G = \langle a, b; a^{-1}b^{12}a = b^{13} \rangle \]

is Hopfian but possesses a normal subgroup of finite index which is non-Hopfian.

It turns out that \( G'' \), the second derived group of

\[ G = \langle a, b; a^{-1}b^2a = b^3 \rangle \]

is free. This fact enables us to prove the following theorem (cf. B H. Neumann [2, p. 544]).

\(^4\) \( G \) is residually finite if for each \( x \in G \) \((x \neq 1)\) there corresponds a normal subgroup \( N_x(G) \) such that \( G/N_x \) is finite and \( x \not\in N_x. \)
Theorem 3. The groups

\[ G = \langle a, b ; a^{-1}b^2a = b^3 \rangle \]

and

\[ H = \langle c, d ; c^{-1}d^3 = d^3, ([c, d]^2c^{-1})^2 = 1 \rangle \]

are homomorphic images of each other; however they are not isomorphic.

Finally we employ Theorem 1 to provide the first instance of a two-generator group which is soluble-of-length-three and non-Hopfian. Thus

Theorem 4. There exists a two-generator group which is soluble-of-length-three and non-Hopfian.

Theorem 4 may be compared with the results of B. H. Neumann and Hanna Neumann [7] and P. Hall [8].

References

4. A. I. Mal’cev, On isomorphic representations of infinite groups by matrices, Mat. Sb. 8 (1940), 405–422.

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