RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

INVARIANT QUADRATIC DIFFERENTIALS

BY JOSEPH LEWITTES

Communicated by R. Bott, March 21, 1962

Let $S$ be a compact Riemann surface of genus $g \geq 2$ and $h$ an automorphism (conformal homeomorphism onto itself) of $S$. $h$ generates a cyclic group $H = \{I, h, \cdots, h^{N-1}\}$ where $N$ is the order of $h$. We shall assume that $N$ is a prime number. Let $D_m$ for an integer $m \geq 0$ denote the space of meromorphic differentials on $S$ and $A_m \subset D_m$ the subspace of finite analytic (without poles) differentials. We obtain representations of $H$ by assigning to $h$ the linear transformation of $D_m$ into itself by $h(\mathcal{D}) = \mathcal{D}^h$ for every $\mathcal{D} \in D_m$. It is clear that $h$ takes $A_m$ into itself so that by restricting to $A_m$ we have a representation of $H$ by a group of linear transformations of a finite dimensional vector space.

In this note we are concerned with determining some of the properties of $(h)$, the diagonal matrix for $h$, considering $h$ as a linear transformation on the $3g-3$ dimensional space $A_2$ of quadratic differentials. Since $(h)^N = (I)$ it is clear that each diagonal element of $(h)$ is an $N$th root of unity. If $e^k$ is an $N$th root of unity, denote by $n_k$ the multiplicity of $e^k$ ($k = 0, 1, \cdots, N-1$) in $(h)$.

Let $\hat{S} = S/H$ be the orbit space of $S$ under $H$. Then it is well known that $\hat{S}$ can be given a conformal structure and the projection map $\pi: S \rightarrow \hat{S}$ is then analytic. The branch points of this covering are precisely at the $t$ fixed points of $h$, $P_1, \cdots, P_t \in S$, $t \geq 0$—here we make essential use of the assumption that $N$ prime—each a branch point of order $N-1$. Let $g_1$ be the genus of $\hat{S}$. The Riemann-Hurwitz formula reads $2g - 2 = N(2g_1 - 2) + (N-1)t$. Now clearly $n_0$ is the dimension of that subspace of $A_2$ which consists of $H$—invariant differentials, i.e., those satisfying $h(\mathcal{D}) = \mathcal{D}$.

**Theorem 1.** (i) $n_0$, the dimension of the space of $H$-invariant finite quadratic differentials, is $3g_1 - 3 + t$.

(ii) If $n_k \neq 0$ for some $k$, $1 \leq k \leq N-1$, then

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1 This is a brief edited excerpt from my thesis submitted to Yeshiva University, 1962.
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(*) \[
3g_1 - 3 + 2 \frac{(N - 1)}{N} t \geq n_k \geq 3g_1 - 3 + \frac{(N - 1)}{N} t.
\]

(iii) There exists \( k^* \), \( 1 \leq k^* \leq N - 1 \), for which \( n_k \neq 0 \).

(iv) If \( g_l \geq 1 \) then \( n_0 \leq 3g_1 - 3 \) unless \( S \) is a surface with equation \( y^2 = x^4 + Ax^4 + Bx^2 + 1 \), in which case \( g = 2, g_l = 1, n_0 = 2 = 3g - 4 = 3g_1 - 3 + t \).

The proof of (i) is similar to the proof of (ii) given below. (iii) follows immediately from the

**LEMMMA.** The representation \( h^m \rightarrow (h)^m \), \( m = 0, 1, \ldots N - 1 \), of \( H \) is faithful, i.e., \( (h)^m = (I) \) implies \( m = 0 \) unless \( g = 2 \) and \( h = J \), the hyperelliptic involution.

The simple proof of this lemma is in my thesis and is omitted here.

(iv) is an immediate consequence of (iii) and (ii) since (*) then implies \( n_k \geq 2 \) unless \( g_1 = 1, N = 2, t = 2 \) (\( g_1 = 1 \) implies \( t \geq 1 \) by Riemann-Hurwitz) or \( g_1 = 1, t = 1 \). But if \( g_1 = 1, t = 1 \) then by (i) one has \( n_0 = 1 \leq 3g - 5 \) for \( g \geq 2 \). The first exception is the case indicated in (iv) with \( h : x \rightarrow -x, y \rightarrow y \) and fixed points on the two sheets over \( x = 0 \).

To prove (ii) let \( \theta \in A_2 \) be such that \( h(\theta) = e^{\theta} \). At any fixed point \( P \in \{ P_1, \ldots P_t \} \) say \( h^{-1} : z \rightarrow \eta z \) in terms of a suitable local parameter, \( \eta^N = 1, \eta \neq 1 \). Then we must have \( \theta h^{-1} = (a_0 + a_1(\eta z) + \ldots )\eta^2 dz^2 = e^\theta (a_0 + a_2z + \ldots )dz^2 \). Thus \( a_n = 0 \) unless \( n + 2 \equiv l \) (mod \( N \)) where \( \eta^l = e^\theta; 1 \leq l \leq N - 1 \). \( \theta \) then actually has an expansion of the form at \( P \) in \( z \),

\[
\theta = (a_{l-2}z^{l-2} + \ldots + a_{kN+l-2}z^{kN+l-2} + \ldots )dz^2
\]

(if \( l \geq 2 \); if \( l = 1 \) the first term must be omitted). This then holds for every \( \theta \) for which \( h(\theta) = e^{k\theta} \). To each point \( P_i, i = 1 \ldots t \), we have then \( \eta_i^l = e^\theta \), for suitable \( \eta_i, l_i \). Such a \( \theta \) then necessarily has at \( P_i \) a zero of the form \( r_i N + l_i - 2 \geq 0 \) and the divisor of \( \theta \) must be of the form \( (P_i^{N+l_i-1}Q_i^{m_i}h(Q_i)^{m_i} \ldots h^{N-1}(Q_i)^{m_i}) \) where the \( Q_i \) are nonfixed points of \( h \).

Let us partition the \( P_i \) into \( P_1 \ldots P_u \), and \( P_{u+1} \ldots P_t \), \( 0 \leq u \leq t \), where \( P_i \) for \( i \leq u \) has \( l_i = 1 \) and \( P_i \) for \( i > u \) has \( l_i \geq 2 \). If \( h(\phi) = e^{i\phi} \) also, then \( \phi/\theta = f \) is an \( H \) invariant function on \( S \) which may be con­strued as a function \( f \) on \( \bar{S} \). Then, since \( \delta \theta \), for fixed \( \theta \) and \( f \) varying over all \( H \) invariant functions with poles at most at the zeros of \( \theta \), gives us all differentials \( \phi \in A_2 \) for which \( h(\phi) = e^{i\phi} \), we have to compute the dimension of this space of functions on \( \bar{S} \). At a point \( P_i, i \leq u, \phi/\theta = f \) is
but \( r'_i \) is at least 1, so that \( f \) has a pole of order at most \((r_i - 1)N\).

On the other hand, at \( P_i, i > u, \phi/\theta = f \) is \( z^{-(r_i - r'_i)N} \) where \( r'_i \) may be 0, so that \( f \) may have a pole of order at most \( r_i N \). Thus, on \( \hat{S} \), \( f \) must be a multiple of the divisor

\[
\omega = (p_1^{1-r_1} \cdots p_{u+1}^{1-r_{u+1}} \cdots p_t^{1-r_t})^{\omega^{-1}}).
\]

We now have \( n_\ell = \text{deg } (\omega^{-1}) + i(\omega^{-1}) + 1 - g_1 \). The degree of the divisor \( (\theta) \) is

\[
4g - 4 = \sum_{i=1}^t (r_i N + l_i - 2) + N \sum m_j
\]

\[
= N \left( \sum_{i=1}^t r_i + \sum m_j \right) + \sum_{i=u+1}^t (l_i - 2) - u.
\]

Therefore,

\[
\text{deg } (\omega^{-1}) = \sum_{i=1}^t r_i - u + \sum m_j = \frac{4g - 4 - \sum_{i=u+1}^t (l_i - 2) - (N - 1)u}{N}.
\]

This is as small as possible when \( u = t \) and as large as possible when \( u = 0 \) and each \( l_i = 2 \). When \( u = t \) we have \( \text{deg } (\omega^{-1}) = (4g - 4)/N - ((N - 1)/N)t \). Using the Riemann-Hurwitz relation gives, \( \text{deg } (\omega^{-1}) = 4g_1 - 4 + ((N - 1)/N)t > 2g_1 - 2 \), so that \( i(\omega^{-1}) = 0 \) in any event. When \( u = 0 \) and each \( l_i = 2 \), we have \( \text{deg } (\omega^{-1}) = (4g - 4)/N = 4g_1 - 4 + 2((N - 1)/N)t \). This completes the proof of (ii).