MARKOV PROCESSES WITH IDENTICAL HITTING DISTRIBUTIONS

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1. Introduction. Throughout \( X \) and \( X^* \) are to be time homogeneous Markov processes taking values in a locally compact, noncompact, separable metric space \( E \), and both satisfying Hunt’s condition (A) \([2, \text{pp. 48–50}]\). The purpose of this note is to give rather general conditions under which there exists a continuous random time change \( \tau(t) \), in the sense of \([4, \text{p. 104}]\), such that \( X(\tau(t)) \) and \( X^*(t) \) are equivalent, that is that they have the same transition function. Obviously a necessary condition, at least if \( \tau(t) \to \infty \) as \( t \to \infty \), is that the two processes have the same hitting distributions in the sense of hypothesis (hi) below. Our theorem is that under a mild additional assumption this condition is also sufficient. A full proof will be published elsewhere.

2. Hypotheses. Let \( P(t, x, A) \) be the transition function for the process \( X \), \( P_x \) and \( E_x \) the probabilities and expectations for \( X \) starting at \( x \), \( T_A \) the infimum of the strictly positive \( t \) such that \( X(t) \) is in the subset \( A \) of \( E \) and \( H_A(x, B) = P_x(X(T_A) \in B, T_A < \infty) \), \( A \) and \( B \) being Borel sets. Analogous quantities for \( X^* \) are denoted by \( P^*, E^*, T^* \) and \( H^* \) with appropriate arguments. Our hypotheses are these: (hi) for each \( x \) in \( E \) and compact \( K \), \( H_K(x, \cdot) = H^*_K(x, \cdot) \), and (h₂) there is an increasing sequence \( \{G_n\} \) of compact sets whose union is \( E \) and such that, for each \( x \) and \( n \), \( P_x(T_{G_n} < \infty) = 1 \). The \( c \) here denotes complement.

3. Outline of construction. Fix one of the sets \( G = G_n \) and suppress the subscript. If \( f_\lambda(x) = E_x^\ast(1 - \exp(-\lambda T_{G_n}^\ast)) \), \( \lambda > 0 \), then \( f_\lambda \) is excessive for the process \( X^* \) terminated when it first leaves \( G \). By a theorem of Dynkin [1] it is then also excessive for \( X \) similarly terminated. One can show that \( f_\lambda \) is regular enough that arguments of Šur [5] and Volkonskii [6] apply to it and yield a continuous additive functional \( \phi_\lambda(t) \) satisfying \( E_x \phi_\lambda(T_{G_n}^\ast) = f_\lambda(x) \). One next shows that \( \lambda^{-1}\phi_\lambda(t) \) increases, as \( \lambda \to 0 \), to a continuous strictly increasing additive functional which, reintroducing the index \( n \), we call \( \phi^n(t) \). The \( \phi^n \) for varying \( n \) are shown to be compatible in the sense that if \( m > n \) then for

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all $x$ with $P_x$ probability one $\phi^n(t) = \phi^n(t)$ throughout the interval $t < T_\alpha^n$. The limit as $n \to \infty$ of $\phi^n(t)$ is a continuous additive functional $\phi(t)$.

The desired time change $\tau(t)$ is the functional inverse to $\phi$. That $X(\tau(t))$ is equivalent to $X^*(t)$ follows from the computation of certain potentials.

4. Remarks. Usually the hypothesis (h$_2$) may be eliminated. For example if the semi-group for one of the processes leaves invariant the space of bounded continuous functions on $E$ then (h$_1$) alone implies the existence of the desired time change.

In [3] there appears a more explicit form of our result in case $X$ is Brownian motion in Euclidean space and $X^*$ is a diffusion process with the same hitting distributions. The construction makes use of potential theoretic facts which are available for transition functions having a sort of symmetry, but not for those as general as the ones we consider here.

The results announced here are also valid for processes having finite terminal times.

References


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