ON REAL JORDAN ALGEBRAS

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Let $X$ be a vector space of the finite dimension $n$ over the field $\mathbb{R}$ of the real numbers. For a (scalar or vector valued) function $f$ defined in a neighbourhood of $x \in X$ and differentiable in $x$, the operator

$$\Delta^u_f(x) = \left. \frac{d}{d\tau} f(x + \tau u) \right|_{\tau=0}$$

is defined and linear for every $u \in X$.

We consider triples $(Y, \omega, c)$ fulfilling the following conditions:

(i) $Y$ is an open and connected subset of $X$ such that $\lambda > 0$ and $y \in Y$ implies $\lambda y \in Y$.

(ii) $\omega = \omega(y)$ is a continuous real-valued function on the closure $\overline{Y}$ of $Y$ that is homogeneous of degree $n$, positive, and real analytic in $Y$, and vanishes on the boundary of $Y$. Furthermore, the Hessian $\Delta^u \Delta^v \log \omega(y)$ is nonsingular for $y \in Y$.

Let $c$ be a given point in $Y$ and denote by $\sigma(u, v)$ the Hessian of $\log \omega(y)$ at the point $y = c$. Without restriction we may assume that $\omega(c) = 1$ holds. Since $\sigma(u, v)$ is nonsingular, the adjoint transformation $A^*$ (with respect to $\sigma$) is defined for every linear transformation $A$ of $X$. We form the group $\Sigma'$ of those linear transformations $W$ of $X$ for which $y \mapsto Wy$ is a bijective map of $Y$ onto itself and for which $\omega(Wy) = |W|^{1/2} \omega(y)$ holds identically for $y \in Y$. Here $|W|$ denotes the absolute value of the determinant of $W$. Let $\Sigma$ be the subgroup of $\Sigma'$ consisting of the transformations $W$ in $\Sigma'$ for which $W^* \in \Sigma'$ holds. The triple $(Y, \omega, c)$ is called an $\Omega$-domain, if (i), (ii) hold and in addition

(iii) $\Sigma$ acts transitively on $Y$.

On the other hand, we consider in $X$ a Jordan algebra, i.e., a bilinear and commutative composition $(x, y) \rightarrow xy$ of $X \times X \rightarrow X$ fulfilling

$$x^2(xy) = x(x^2y)$$

for every $x, y$ in $X$. Such a Jordan algebra, that is, the vector space $X$ together with the composition, shall be denoted by $A$. For every $x \in X$ the mapping $y \rightarrow xy$ determines a linear transformation $L(x)$ of $X$ such that $xy = L(x)y$. Denote by $\tau(x, y)$ the trace of $L(xy)$. Then $\tau(x, y)$ is a symmetric bilinear form on $X$. The Jordan algebra $A$ is called semi-simple if $\tau(x, y)$ is nonsingular. It is known, that a semi-simple Jordan algebra contains a unit element $c$. Besides the linear
transformation $L(x)$ there is another, more important transformation $P(x)$ introduced by N. Jacobson [1] that is defined by

$$y \rightarrow P(x)y = 2x(xy) - x^2 y,$$

i.e., $P(x) = 2L^2(x) - L(x^2)$. $P(x)$ fulfills the following identity

$$P(P(x)y) = P(x)P(y)P(x), \quad x, y \in X.$$  

A first proof of this formula (in case of semi-simple real Jordan algebras) can be found in Ch. Hertneck [5]. The proof for general Jordan algebras was given independently by I. G. MacDonald [6]. Since $P(c)$ is the identity mapping, the determinant $|JP(x)|$ is not identically zero. The transformation $P(x)$ is called the quadratic representation of the Jordan algebra $A$. Formula (1) shows for instance

$$P(x^m) = P^m(x) \quad \text{for } m = 1, 2, \ldots.$$  

However, $P(x)$ is not a representation in the sense, that $x \rightarrow P(x)$ is a homomorphism of $A$ into the Jordan algebra of linear transformations.

To a given semi-simple Jordan algebra $A$ we define a triple $(YA, O_A, C_A)$, denoted by $Q(A)$, in the following way: $C_A$ is the unit element of $A$, $O_A(x) = |P(x)|^{1/2}$ and $Y_A$ is the connected component of the set $\{x; O_A(x) \neq 0\}$ containing the point $C_A$.

Using certain results on Jordan algebras, in particular formula (1), the theory of eigenvalues of a Jordan algebra (following ideas of E. Artin) and the notion of the inverse in a Jordan algebra (due to N. Jacobson [1], following a representation of E. Artin), we are able to prove

**Theorem 1.** Let $A$ be a semi-simple Jordan algebra over $R$. Then $\Omega(A)$ is an $\Omega$-domain in the vector space underlying $A$. The bilinear form $\sigma$ associated with the $\Omega$-domain coincides with the bilinear form $\tau$ of $A$. Moreover, the transformations $P(x)$, $|P(x)| \neq 0$, belong to the group $\Sigma$ which is associated with the $\Omega$-domain.

In the course of the proof it turns out, that even the group generated by the transformations $P(x)$ where $x$ varies in some neighbourhood of the unitelement $C_A$, acts transitively on $Y_A$.

Vice versa, let us start out with an $\Omega$-domain $(Y, \omega, c)$ in $X$. An investigation of the geodesics with respect to the (in general not positive definite) metric given by the Hessian $\Delta \omega_{ij} \log \omega(y)$ leads to

**Theorem 2.** Let $(Y, \omega, c)$ be an $\Omega$-domain in $X$. Then there exists a semi-simple Jordan algebra $A$ in $X$ such that $(Y, \omega, c) = \Omega(A)$.

Furthermore we get
Theorem 3. The map $A \rightarrow \Omega(A)$ of the family of semi-simple real Jordan algebras $A$ is a bijection onto the family of $\Omega$-domains.

It is important to know under which circumstances two semi-simple Jordan algebras give rise to $\Omega$-domains that are not essentially different. We call two $\Omega$-domains $(Y, \omega, c)$ resp. $(Y', \omega', c')$ defined over the vector space $X$ resp. $X'$ of the same dimension, equivalent if there is a bijective linear transformation $V: X \rightarrow X'$ such that

$$Y' = VY, \quad \omega'(Vy) = \gamma \cdot \omega(y) \quad \text{for } y \in Y$$

holds, where $\gamma$ is a suitable real number. Then we get

Theorem 4. Two $\Omega$-domains $\Omega(A)$ and $\Omega(B)$ are equivalent if and only if the Jordan algebras $A$ and $B$ are isomorphic.

Given a Jordan algebra $A$ and $f \in A$. Then one can define a new multiplication in the underlying vector space $X$ by

$$x \perp y = x(yf) + y(xf) - (xy)f.$$  

$X$ together with the composition $\perp$ shall be denoted by $A_f$. It is known that $A_f$ is a Jordan algebra. The quadratic representation of $A_f$ turns out to be $P(x)P(f)$, where $P(x)$ is the quadratic representation of $A$.

Given a semi-simple Jordan algebra $A$, let us consider the subset $X_A$ of the underlying vector space $X$ consisting of all points $x$ for which $|P(x)| \neq 0$. The connected component of the unitelement of $A$ is an $\Omega$-domain (see Theorem 1). In addition we get

Theorem 5. Let $C$ be a connected component of $X_A$ and $f \in C$, then the triple $(C, |P(f)|^{1/2} \cdot \omega_A, f^{-1})$ is the $\Omega$-domain, which is associated with the semi-simple Jordan algebra $A_f$.

Here $f^{-1}$ denotes the inverse of $f$ in the Jordan algebra $A$. Combining Theorems 4 and 5 we have

Theorem 6. There is a one-to-one correspondence between the equivalence classes of connected components of $X_A$ (considered as $\Omega$-domains) and the isomorphic classes of the Jordan algebras $A_f$ where $f \in X_A$.

Special cases of $\Omega$-domains are the homogeneous domains of positivity (see [2; 3; 4], O. S. Rothaus [7], Ch. Hertneck [5] and E. B. Vinberg [8]). It is known, that the map $A \rightarrow \Omega(A)$ maps the family of formal real Jordan algebras onto the family of homogeneous domains of positivity. Here a Jordan algebra $A$ is called formal real if $x^2 + y^2 = 0$ implies $x = y = 0$. This is equivalent to the notion of a compact Jordan
algebra, i.e., a Jordan algebra, for which the bilinear form \( \tau(x, y) \) is positive definite. This gives an algebraic characterization of the Jordan algebras associated with domains of positivity. However, there is a different geometric characterization of the domains of positivity in the family of \( \Omega \)-domains.

**Theorem 7.** An \( \Omega \)-domain \((Y, \omega, c)\) is an homogeneous domain of positivity if and only if the set \( Y \) is convex.

**References**


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