A NOTE ON THE KO-THEORY OF SPHERE-BUNDLES

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1. Two vector bundles $E$ and $F$ over the finite connected CW complex $X$ are $J$-equivalent, if their sphere-bundles $S(E)$ and $S(F)$ are of the same fiber-homotopy type. If they become $J$-equivalent after a suitable number of trivial bundles is added to both of them they are stably $J$-equivalent.

This note concerns itself with a new stable $J$-invariant $\theta(E)$, which was suggested by the recent work of Atiyah-Hirzebruch [2] and F. Adams [1]. In fact $\theta(E)$ bears the same relation to the Adams operation $\psi_i$, as the Stiefel-Whitney class—a known $J$-invariant—bears to the Steenrod operations.

2. We assume familiarity with the Grothendieck group, $KO(X)$, of real vector bundles over $X$, and with the extension of this functor to a cohomology theory: $KO^*(X) = \bigoplus_{i=-\infty}^{\infty} KO^i(X); KO^0(X) = KO(X)$. Also that the exterior powers,

$$\lambda^i: KO(X) \to KO(X)$$

as defined in [3] are understood. In terms of these the Adams operations $\psi_i$, are defined as follows:

Set $\lambda_i(x) = \sum \lambda^i(x)t^i, \ x \in KO(X), \ \lambda_i(x) \in KO(X)[[t]]$.

Now define $\psi_{-i}(x)$ as the power series:

$$(2.1) \quad \psi_{-i}(x) = - \frac{t^i}{i} \frac{d}{dt} \{ \log \lambda_i(x) \} = - \frac{\lambda_i'(x)}{\lambda_i(x)}$$

and set $\psi_i(x)$ equal to the coefficient of $t^i$ in $\psi_i(x)$.

The familiar formula: $\lambda_i(x+y) = \lambda_i(x) \cdot \lambda_i(y)$ then goes over into $\psi_i(x+y) = \psi_i(x) + \psi_i(y)$ so that the $\psi_i$ are additive and therefore much more tractable than the $\lambda^i$. Note that if $L$ is a line-bundle then by (2.1) $\psi_kL = L^k$. Other important properties of $\psi_i$ are [1]:

$$(2.2) \quad \psi_k \text{ is a ring homomorphism, which commutes with the } \lambda^i.$$

$$(2.3) \quad \psi_k \cdot \psi_s = \psi_{ks}.$$

3. In the $KO^*$ terminology the periodicity theorem for the orthogonal groups [4; 2] states that the map

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induced by the tensor product of bundles is an isomorphism. The following theorem, which is implicitly contained in [2], extends this result:

**Theorem I.** Let $E$ have dimension $(8n+1)$ over $X$, and assume that the structure group of $E$ may be reduced to $\text{Spin}(8n+1)$. Let $\pi: S(E) \to X$ be the associated sphere-bundle and write $\tilde{E}$ for the bundle along the fibers in $S(E)$. The structure group of $\tilde{E}$ then has a reduction to $\text{Spin}(8n)$, and we let $\Delta^+(\tilde{E})$ be the vector bundle associated to $\tilde{E}$ by one of the real Spin-representations of $\text{Spin}(8n)$.

Under these circumstances $KO^*\{S(E)\}$ is a free module over $KO^*(X)$ with generators, 1 and $\Delta^+(\tilde{E})$:

$$KO^*\{S(E)\} = KO^*(X)[y], \quad y = \Delta^+(\tilde{E}).$$

An immediate consequence of Theorem I is that formulae of the type:

$$2^y = A(E)y + B(E)y = \Delta^+(\tilde{E}).$$

must hold, and thus define four invariants of $E$ in $KO(X)$. The $\theta(E)$ will be fashioned from the $\theta_k(E)$.

Remark first that a fiber homotopy equivalence $f$, between $S(E)$ and $S(F)$ induces a $KO^*(X)$-homomorphism, $f^*$ of $KO^*\{S(F)\}$ onto $KO^*\{S(E)\}$.

Hence:

**Theorem II.** The bundles $E$ and $E'$ subject to the condition of Theorem I are $J$-equivalent only if:

$$\theta_k(E) = \theta_k(E') \cdot \psi_k u/u, \quad k \in \mathbb{Z}^+$$

where $u \in KO(X)$ is an invertible element, i.e. $\text{dim } u = 1$.

We have in addition: (Adams [1], see also (4.7)).

**Proposition 3.1.** When $E$ is the trivial bundle, then

$$\theta_k(E) = k^4n.$$ 

One may now clearly formalize the stable instance of (3.4) and (3.5) as follows:

**Definition.** A function $f: \mathbb{Z}^+ \to KO(X)$ is called a cocycle if

$$f(ts) = \{\psi_t f(s)\} f(t), \quad s, t \in \mathbb{Z}^+,$$
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(3.7) \( \dim f(t) = t^{4n(f)} \),

where \( n(f) \) is a fixed integer \( \in \mathbb{Z}^+ \).

The cocycles form a multiplicative system, \( f \cdot g(t) = f(t) \cdot g(t) \). Two cocycles \( f, g \) are called equivalent, if there exist integers \( n, m \in \mathbb{Z}^+ \), and an element \( u \in KO(X) \) with \( \dim u = 1 \), so that:

\[ t^{4n} f(t) = \left\{ t^{4m} \left( \psi u \right) / u \right\} \cdot g(t), \quad t \in \mathbb{Z}^+. \]

The equivalence classes are then seen to form an abelian group which is denoted by \( H^1(\mathbb{Z}^+; KO(X)) \).

Consider now the bundle \( E \) of our discussion. By (3.3) and (3.5) the function \( t \rightarrow \theta(E) \) determines a cocycle and hence an element \( \theta(E) \in H^1(\mathbb{Z}^+; KO(X)) \).

**Theorem III.** The element \( \theta(E) \) is a stable \( J \)-invariant of \( E \). Further, one has \( \theta(E \oplus E' \oplus 7 \cdot 1) = \theta(E) + \theta(E') \), so that \( \theta \) extends to a homomorphism

\[ \theta: K \text{Spin}(X) \rightarrow H^1(\mathbb{Z}^+; KO(X)). \]

Here of course \( K \text{Spin}(X) \) is the subring of \( KO(X) \) generated by vector bundles which admit a Spin-reduction.

It is customary to denote the stable \( J \)-classes of vector-bundles by \( J(X) \). If we define \( \Theta(X) \) as the image of \( \theta: K \text{Spin}(X) \rightarrow H^1(\mathbb{Z}^+; KO(X)) \), then Theorem III shows that \( \Theta(X) \) furnishes a lower bound for \( J(X) \) in the sense that \( \theta \) induces a surjection of a subgroup of \( J(X) \) onto \( \Theta(X) \).

4. We have seen that the Theorems II and III are really quite straightforward consequences of Theorem I. The question now arises how the invariants \( \theta_k(E) \), etc., are to be computed. For instance can they be expressed in terms of the \( \lambda^i(E) \)?

**Theorem IV.** Let \( E \) be as in Theorem I and set \( \Delta(E) \in KO(X) \) equal to the bundle which the spin representation of \( \text{Spin}(8n+1) \) associates to \( E \), via the Spin reduction of \( E \).

Then

\[ \{ \Delta(E) \}^2 = \frac{1}{2} \lambda_1(E) \quad \text{in KO}(X). \]

(4.2) The invariants \( A, B, \theta_k, \Gamma_k \), of \( E \) are given by universal polynomials in \( \Delta(E) \), and the \( \lambda^i(E) \).

Particular examples are as follows:

(4.3) Both, \( A(E) \) and \( \theta_2(E) \) are equal to \( \Delta(E) \),
(4.4) \[ B(E) = - \sum_{i=1}^{2n-1} \lambda^{2i-1}(E - 1), \]

(4.5) \[ \theta_{2k+1}(E) \text{ is a polynomial in the } \lambda^i E. \]

In general the expression for \( \theta_k \) is quite complicated as the following recipe for \( \theta_k \) shows:

Algorithm. Consider the ring of finite Laurent-series \( L = \mathbb{Z}[x_i, x_i^{-1}], \)
\( i = 1, \ldots, 4n. \) Define \( \gamma^i, \omega, \) and \( \eta_k \) in \( L \) by:

\[ \sum_{i} \gamma^i = (1 + t) \prod_{i} (1 + tx_i)(1 + tx_i^{-1}), \]
\[ \omega = \prod_{i} (x_i + x_i^{-1}), \]
\[ \eta_k = \prod_{i} (x_i^k + \cdots + x_i^{-k}). \]

Express \( \eta_k \) as a polynomial in the \( \gamma^i \), and \( \omega \):

\[ \eta_k = P_k(\gamma^i, \omega), \]

then

\[ \theta_k(E) = P_k(\lambda^i(E); \Delta(E)). \]

In special circumstance \( \theta_k \) can of course be computed much more simply. As an example we cite:

**Proposition 4.1.** If \( E = 8nL + 1 \), where \( L \) is a line bundle, then:

(4.6) \[ \theta_k(E) = (1 + L + \cdots + L^{k-1}) \]

whence

(4.7) \[ \theta_k(E) = \begin{cases} k^{4n} + \frac{k^{4n}}{2}(L - 1) & k \text{ even}, \\
     k^{4n} + \frac{(k^{4n} - 1)/2}{2}(L - 1) & k \text{ odd}. \end{cases} \]

Note that (4.6) and (4.7) imply Proposition 3.1.

The general idea behind these computations is the following one:

If \( G \) is a Lie group, we write \( RO(G) \) for the character ring (Grothendieck ring) of the finite dimensional \( G \) modules over \( \mathbb{R} \) (see [2]).

The exterior powers \( \lambda^i \), and hence the \( \psi_i \) also may be introduced into these rings just as they were introduced into \( KO(X) \). From representation theory we learn that:

**Proposition 4.2.** If we write \( G \) for \( \text{Spin}(8n+1) \), \( H \) for \( \text{Spin}(8n) \) and let \( i: H \to G \) be the standard inclusion. Then
(4.8) \( i^*: \text{RO}(G) \to \text{RO}(H) \) is an injection,

(4.9) \( \text{RO}(H) \) is a free module over \( \text{RO}(G) \) generated by the unit element and the spin representation \( \Delta^+ \),

(4.10) \( \text{RO}(G) \) is a polynomial ring with generators \( \rho, \lambda \rho, \cdots, \lambda^{4n} \rho, \Delta \)

where \( \rho \) is the standard \((8n+1)\) dimensional \( G \)-module and \( \Delta \) is the spin-representation.

**Corollary.** There are unique elements \( A, B, \theta_k, \Gamma_k \in \text{RO}(G) \) such that

\[
(\Delta^+)^2 = \Delta^+ \otimes i^*A + i^*B,
\]

(4.11) \[
\psi_k(\Delta^+) = \Delta^+ \otimes i^*\theta_k + i^*\Gamma_k.
\]

Further these are well determined polynomials in \( \lambda^i \rho, \Delta \).

The similarity of these formulae to (3.3) is clear, and in fact the corollary yields Theorem IV by virtue of the following two quite general facts:

1. If \( G \) is a Lie group and \( \theta \) an element in \( \text{RO}(G) \), then \( \theta \) defines a functor \( \xi \to \theta(\xi) \) from principal \( G \)-bundles over \( X \) into \( \text{KO}(X) \): if \( \theta \) is an “actual” \( G \)-module \( \theta(\xi) \) is the associated vector bundle.

2. Suppose now that \( \xi \) has total space \( Y_\xi \) and that \( \iota: H \to G \) is a closed subgroup of \( G \). Then \( Y_\xi \to Y_\xi/H \) defines \( Y_\xi \) as a principal \( H \) bundle, \( \hat{\xi} \), over \( Y_\xi/H \). Consider the \( G/H \)-bundle \( \pi: Y_\xi/H \to X \). Under these circumstances the following identity holds in \( \text{KO}(Y_\xi/H) \).

**Permanence Law.** Let \( \alpha \in \text{RO}(H), \beta \in \text{RO}(G) \). Then

\[
(\alpha \otimes i^*\beta)(\hat{\xi}) = \alpha(\hat{\xi}) \otimes \pi^*\beta(\xi) \quad \text{in} \ \text{KO}(Y_\xi/H).
\]

**Remarks.** 1. The invariant \( \theta \) seems to yield the best presently known information about \( J(X) \). For instance with the aid of (4.3) and (4.1) one may compute \( \Theta \{ S_{4k} \} \) and obtains a cyclic group of order equal to the denominator of the \( k \)th Bernoulli number over \( 4k \). For \( \Theta \{ S_n \}, n = 1, 2 \mod 8 \) one finds (via (4.1)) the group \( \mathbb{Z}_2 \). Finally this same formula leads to the result that for real projective space \( \tilde{K}O(P_n) \simeq J(P_n) \). This beautiful result of Adams is the central step in his solution of the vector-field problem on spheres.

2. Completely analogous formulae hold for the \( KU \) theory under more general circumstances. (\( E \) need only admit a reduction to \( \text{Spin}(2n+1) \times \mathbb{Z}_2^* \).)

3. The operation of \( \lambda^i \) in \( \text{KO}^*\{ S(E) \} \) could be determined in an
analogous fashion but lead to very messy formulae, which furthermore give no additional stable information.

4. Finally a word concerning the proof of Theorem I. It is a known result that when \( X = \text{point} \), then \( KO \{ S(E) \} = KO(S^{8n}) \) is generated by 1 and \( y \). (See [2]). Hence (2.1) proves the first statement of Theorem I whenever \( E \) is trivial. Now an inductive Meyer-Vietoris argument yields the general case.

BIBLIOGRAPHY


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A CONNECTION BETWEEN TAUBERIAN THEOREMS AND NORMAL FUNCTIONS

BY G. T. CARGO

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The purpose of this note is to point out that certain Tauberian theorems follow immediately from some recent research of Lehto-Virtanen and Bagemihl-Seidel.

Let \( D \) denote the open unit disk, let \( C \) denote the unit circumference, and let \( \rho(z_1, z_2) \) denote the non-Euclidean hyperbolic distance between the points \( z_1 \) and \( z_2 \) in \( D \).

THEOREM. Suppose that \( f(z) = \sum a_nz^n \) and that \( n |a_n| \leq M \) \((n = 1, 2, \cdots)\) for some constant \( M \). Further, suppose that \( \{z_n\} \) is a sequence of points in \( D \) converging to a point \( \xi \) in \( C \) with the property that \( \rho(z_n, z_{n+1}) \to 0 \) as \( n \to \infty \). Then, if \( f(z_n) \to c \) as \( n \to \infty \), the series \( \sum a_nz^n \) converges to the sum \( c \).

PROOF. The hypothesis implies that \( |f'(z)| \leq M/(1-|z|) \). Consequently, \( \rho(f(z))|dz| \leq 2M\sigma(z) \) holds for all \( z \) in \( D \) where \( \rho(f(z)) = |f'(z)|/(1+|f(z)|^2) \) denotes the spherical derivative of \( f \) and \( d\sigma(z) \)

\[ 1 \text{ National Academy of Sciences-National Research Council Postdoctoral Research Associate on leave from Syracuse University.} \]