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SOME CONVOLUTION ALGEBRAS OF MEASURES ON [1, ∞) AND A REPRESENTATION THEOREM FOR LAPLACE-STIELTJES TRANSFORMS

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1. Introduction. In [1] we studied the set \( \mathcal{A}_R \) of power series \( \sum_{n=1}^{\infty} a_n z^n \) convergent for \( |z| < R, 0 < R \leq 1 \), under the multiplication

\[
\left( \sum_{n=1}^{\infty} a_n z^n \right) \left( \sum_{n=1}^{\infty} b_n z^n \right) = \sum_{n=1}^{\infty} \left( \sum_{rs=n} a_r b_s \right) z^n.
\]

It was found that \( \mathcal{A}_R \), with the usual addition and scalar multiplication, and with the topology of uniform convergence on compact subsets of the disk \( |z| < R \), is a locally convex algebra with identity. Also \( \sum_{n=1}^{\infty} a_n z^n \) is invertible (has an inverse in \( \mathcal{A}_R \) with respect to the above multiplication) if and only if \( a_1 \neq 0 \). As a consequence we obtained the following expansion theorem for analytic functions (E. Hille [2]).

**Theorem.** Let \( f(z) \) be analytic for \( |z| < R, 0 < R \leq 1 \), with \( f(0) = 0 \). Then associated with any function \( g(z) \) analytic in \( |z| < R \) with the properties \( g(0) = 0, g'(0) \neq 0 \), there is a unique expansion of the form

\[
f(z) = \sum_{n=1}^{\infty} c_n g(z^n), \quad |z| < R.
\]

Our object in this paper is to obtain an analogous result for Laplace-Stieltjes integrals (Theorem 1 below). We shall base the discussion on the theory of convolution algebras of complex measures on \([0, \infty)\).
as described in [3]. More precisely, we shall need the adaptation of this theory to the multiplicative semi-group \([1, \infty)\).

2. Convolution algebras depending on a weight function. In this section we record as Proposition 1 the appropriate modifications of the needed portions of [3].

**Proposition 1.** Let \(\phi(t)\) be a real-valued Borel measurable function defined on \([1, \infty)\) satisfying

\[
0 < \phi(t_1 t_2) \leq \phi(t_1) \phi(t_2), \quad t_1, t_2 \geq 1; \quad \phi(1) = 1.
\]

Let \(\mathcal{B}\) be the ring of bounded Borel subsets of \([1, \infty)\), and let \(S(\phi)\) denote the set of complex measures \(a\) on \(\mathcal{B}\) such that

\[
\left\| a \right\| = \int_1^\infty \phi(t) d\left\| a \right\| (t) < \infty.
\]

Finally let

\[
(a b)(B) = [a \times b]((x, y) \mid xy \in B, \ x \geq 1, \ y \geq 1);
\]

\[
a, b \in S(\phi), \ B \in \mathcal{B}.
\]

Then \(S(\phi)\) is a commutative Banach algebra with norm defined by (2), with multiplication defined by (3), with the obvious definitions of addition and scalar multiplication, and with identity defined by a unit mass at 1. Let

\[
\omega = \inf\{\log \phi(t) / \log t \mid t > 1\}.
\]

If \(\omega = -\infty\), then \(a\) is invertible if and only if \(a(\{1\}) \neq 0, \ a \in S(\phi)\).

**Proof.** Let \(\mathcal{B}'\) be the ring of bounded Borel subsets of \([0, \infty)\). Let the function \(\phi'\) be defined by \(\phi'(t) = \phi(e^t), \ t \geq 0\). Then \(\phi'\) satisfies the requirements given in [3] for a weight function for the additive semi-group \([0, \infty)\). Hence the set \(S'(\phi')\) of complex measures on \(\mathcal{B}'\) with finite \(\phi'\)-norms is a commutative Banach algebra with identity defined by a unit mass at 0. The proof of this statement, given in [3], can readily be adapted to \(S(\phi)\). Let us, however, note that \(S'(\phi')\) and \(S(\phi)\) are isomorphic. Indeed, the exponential function provides an isomorphism of the underlying semi-groups with preservation of bounded Borel sets \(B' \leftrightarrow B\). This induces the one to one correspondence of measures

\[
a' \leftrightarrow a, \ a'(B') = a(B); \ a' \in S'(\phi'), \ a \in S(\phi), \ B' \in \mathcal{B}', \ B \in \mathcal{B},
\]

and this correspondence preserves addition, scalar multiplication, convolution, total variation, and norm. Therefore \(S(\phi)\) is a Banach
algebra as described. To prove the last assertion we note that
\( \omega = \inf \{ \log \phi'(t)/t \mid t > 0 \} \). The assertion now follows from Theorem 4.18.5 of [3] and the isomorphism (5).

3. The special case \( \phi_\sigma(t) = e^{-\sigma(t-1)}, \sigma > 0 \), and Laplace-Stieltjes transforms. We have for \( t_1, t_2 \geq 1 \)
\[
\phi_\sigma(t_1t_2) = e^{-\sigma(t_1t_2-1)} \leq e^{-\sigma(t_1+t_2-2)} = \phi_\sigma(t_1)\phi_\sigma(t_2),
\]
and it is clear that \( \phi_\sigma \) is a suitable weight function for \([1, \infty)\). Furthermore
\[
\log \phi_\sigma(t)/\log t = -\sigma(t-1)/\log t \to -\infty \text{ as } t \to \infty.
\]

Therefore we have the following result.

**Proposition 2.** \( S(\phi_\sigma) \) is a Banach algebra of the type described in Proposition 1. The invertible elements \( a \) of \( S(\phi_\sigma) \) are characterized by the condition \( a(\{1\}) \neq 0 \).

We can now establish the representation theorem alluded to in the Introduction.

**Theorem 1.** Let \( \sigma > 0 \). Let
\[
(6) \quad f(s) = \int_1^\infty e^{-st}da(t), \quad g(s) = \int_1^\infty e^{-st}db(t), \quad \Re s \geq \sigma,
\]
where \( a \) and \( b \) are complex measures on \( \mathfrak{M} \), and the integrals are absolutely convergent. Then
\[
(7) \quad e^{s}g(s) \to b(\{1\}) \text{ as } \Re s \to \infty.
\]
If this limit is not zero, there is a complex measure \( c \) on \( \mathfrak{M} \) such that
\[
(8) \quad f(s) = \int_1^\infty g(st)dc(t), \quad \Re s \geq \sigma,
\]
the integral converging absolutely.

**Proof.** For \( \Re s \geq \sigma \) we have \( e^{s}g(s) = \int_1^\infty e^{-\sigma(t-1)}db(t) \). But \( e^{-\sigma(t-1)} \to \chi_{\{1\}}(t) \) as \( \Re s \to \infty \). Thus we obtain (7) by Lebesgue's dominated convergence theorem. By (6), \( a, b \in S(\phi_\sigma) \), and by Proposition 2 and our assumption regarding (7), \( b \) is invertible. Hence there is a unique \( c \in S(\phi_\sigma) \) such that \( a = bc \). From this equation and the basic definition (3) we conclude
\[
\int_1^\infty e^{-st}da(t) = \int_1^\infty \int_1^\infty e^{-st}d[b \times c](x, y), \quad \Re s \geq \sigma.
\]
By the Fubini theorem this equation implies (8).

4. Remarks. If the integrals in (6) are absolutely convergent in an open half-plane Re \( s > \rho, \rho \geq 0 \), then for every \( \sigma > \rho \) there is a measure \( \mathcal{C}_\sigma \) such that (8) holds. To show that \( \mathcal{C}_\sigma \) is independent of \( \sigma \) we reason as follows. \( a \) and \( b \) belong to the set of measures \( \mathcal{S}_\rho = \bigcap_{\sigma > \rho} \mathcal{S}(\phi_\sigma) \).
In each algebra \( \mathcal{S}(\phi_\sigma), \sigma > \rho, b \) has an inverse, but since these algebras are linearly ordered by inclusion, all these inverses are the same. Thus \( b^{-1} \) exists in \( \mathcal{S}_\rho \). But then \( c = b^{-1}a \) also belongs to \( \mathcal{S}_\rho \). Hence we have a single formula (8) holding in the given half-plane Re \( s > \rho \). The set \( \mathcal{S}_\rho \) is the analog of the set \( \mathcal{A}_R \) of power series. Like \( \mathcal{A}_R \), \( \mathcal{S}_\rho \) is a complete, countably-normed algebra.

In the representation (8) of \( f(s) \), we have regarded \( g \) as having been given, and \( c \) as having been determined. We can reverse the situation and choose any \( c \) subject to the condition \( c(\{1\}) \neq 0 \). The equation \( a = bc \) is then uniquely solvable for \( b \) in the appropriate algebra. (8) now follows as before, \( g \) being the transform of \( b \).

References


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