INERTIA THEOREMS FOR MATRICES: THE SEMI-DEFINITE CASE

BY DAVID CARLSON AND HANS SCHNEIDER

Communicated by A. S. Householder, June 4, 1962

1. The inertia of a square matrix $A$ with complex elements is defined to be the integer triple $\text{In} = (\pi(A), \nu(A), \delta(A))$, where $\pi(A)\{\nu(A)\}$ equals the number of eigenvalues in the open right \{left\} half plane, and $\delta(A)$ equals the number of eigenvalues on the imaginary axis. The best known classical inertia theorem is that of Sylvester: If $P > 0$ (positive definite) and $H$ is Hermitian, then $\text{In} P H = \text{In} H$. Less well known is Lyapunov's theorem [2]: There exists a $P > 0$ such that $\sigma(AP) = \frac{1}{2}(AP + PA^*) > 0$ if and only if $\text{In} A = (n, 0, 0)$. Both classical theorems are contained in a generalization (Taussky [4], Ostrowski-Schneider [3]) which we shall call the

**MAIN INERTIA THEOREM.** For a given $A$, there exists a Hermitian $H$ such that $\text{In} (AH) \geq 0$ if and only if $\delta(A) = 0$. If $\sigma(AH) > 0$, then $\text{In} A = \text{In} H$.

2.1. In this note we consider the case $\sigma(AH) \leq 0$ which is far more complicated than the case $\sigma(AH) > 0$. We do not here solve the problem of all the possible relations of $\text{In} H$ to $\text{In} A$, except under additional assumptions.

**Theorem 1.** Let $A$ be a given matrix for which all elementary divisors of imaginary eigenvalues are linear. If $H$ is a Hermitian matrix such that $\sigma(AH) \geq 0$, then $\pi(H) = \pi, \nu(H) = \nu$ satisfy

(1) $\pi \leq \pi(A) + \delta(A), \quad \nu \leq \nu(A) + \delta(A),$

respectively, and

(2) $\text{rank} \sigma(AH) \leq \pi(A) + \nu(A)$. 

Further, for any triple $(\pi, \nu, \delta)$ for which $\pi + \nu + \delta = n$, and $\pi, \nu$ satisfy (1), there exists an $H$ for which $\sigma(AH) \geq 0$ and $\text{In} H = (\pi, \nu, \delta)$. Thus (1) is in a sense the best possible inequality.

A more precise result may be proved if $\text{rank} \sigma(AH) = \pi(A) + \nu(A)$. 

2.2. Theorem 2 concerns a matrix consisting of just one Jordan
block with one imaginary root. Its proof is largely computational. For assertion (4) (below) we use Cauchy's theorem on the separation of eigenvalues of a Hermitian matrix by the eigenvalues of a principal minor.

**Theorem 2.** Let \( A = \alpha I + U \), where \( \alpha \) is pure imaginary and \( U \) is the matrix with 1 in the first superdiagonal and 0 elsewhere. If \( H \) is Hermitian of rank \( r \) and \( K = \Re(AH) \geq 0 \) is of rank \( s \), then

\[
(3) \quad 2s \leq r,
\]

and for \( \pi(H) = \pi, v(H) = v \),

\[
(4) \quad |\pi - v| \leq 1,
\]

\[
(5) \quad h_{ij} = 0 \quad \text{if } i + j > r + 1,
\]

\[
(6) \quad k_{ij} = 0 \quad \text{if } i > r/2.
\]

Again, the inequalities (3) and (4) are best possible, in the sense that if \( r, s, \pi, v \), with \( \pi + v = r \), are non-negative integers satisfying (3) and (4) then we can find an \( H \) such that \( \Re(AH) \geq 0 \), and \( r = \text{rank } H, s = \text{rank } \Re(AH), \pi = \pi(H) \) and \( v = v(H) \).

As a corollary of Theorem 2 we obtain a general existence theorem:

**Corollary.** For any matrix \( A \), there exists a nonsingular Hermitian \( H \) such that \( \Re(AH) \geq 0 \).

In the special case that all elementary divisors of imaginary roots are linear, this result is known; cf. Givens [1].

2.3. **Theorem 3.** Let \( A \) be a given matrix. If \( H \geq 0 \) and \( \Re(AH) \geq 0 \), then

\[
(7) \quad \text{rank } H \leq \pi(A) + p(A),
\]

where \( p(A) \) is the number of elementary divisors of imaginary roots. The inequality (7) is best possible.

**Corollary 1.** For a given matrix \( A \), there exists an \( H > 0 \) for which \( \Re(AH) \geq 0 \) if and only if

\[
(8) \quad v(A) = 0,
\]

all elementary divisors of imaginary eigenvalues of \( A \) (if any) are linear.

**Corollary 2.** If \( \Re(A) \geq 0 \) and \( H > 0 \) then all elementary divisors of imaginary eigenvalues of \( AH \) are linear.

When \( H = I \), Corollary 2 reduces to part of Theorem 2 of [3].
3.1. The proof of the Main Inertia Theorem hinges on the following lemma: If \( \sigma(AB) > 0 \), then \( B \) is nonsingular. In this section we shall obtain a generalization of the Main Theorem by considering matrices with fixed null-space \( \mathfrak{N} \). By \( \mathfrak{N}(A) \) we shall denote the null-space of \( A(x \in \mathfrak{N}(A): Ax = 0) \) and \( \mathfrak{N}^\perp \) will be the orthogonal complement of \( \mathfrak{N}(x \in \mathfrak{N}^\perp: y^*x = 0 \text{ for all } y \in \mathfrak{N}) \). Our results depend on the easily proved Theorem 4 which takes the place of the lemma quoted above.

We define \( \text{In} A \leq \text{In} B \) if \( \pi(A) \leq \pi(B) \) and \( \nu(A) \leq \nu(B) \) (\( A, B \) need not be of the same order), and \( \text{In} A = \text{In} B \) if \( \pi(A) = \pi(B) \) and \( \nu(A) = \nu(B) \).

**Theorem 4.** If \( \sigma(AB) \geq 0 \) then
\[
\begin{align*}
\mathfrak{N}(\sigma(AB)) & \supseteq \mathfrak{N}(B), \\
A \mathfrak{N}(H)^\perp & \subseteq \mathfrak{N}(H)^\perp, \\
\text{In}(A \mid \mathfrak{N}(H)^\perp) & \leq \text{In} H.
\end{align*}
\]
Here \( A \mid \mathfrak{N}(H)^\perp \) is the restriction of \( A \) to \( \mathfrak{N}(H)^\perp \).

As an immediate corollary to the proposition we have

**Corollary.** If \( \sigma(AB) \geq 0 \) and \( \text{In} (A \mid \mathfrak{N}(H)) = (0, 0, \delta) \) then
\[
\text{In} A = \text{In}(A \mid \mathfrak{N}(H)^\perp) \leq \text{In} H.
\]

*In particular if \( \sigma(AB) \geq 0 \) and \( H \) is nonsingular, then \( \text{In} A \leq \text{In} H \).*

3.2. It is interesting to note that in our next theorem, the inequalities will go in the opposite direction. This theorem reduces to the Main Inertia Theorem when \( \mathfrak{N} = (0) \).

**Theorem 5.** Let \( \mathfrak{N} \) be a subspace of \( V \). There exists a Hermitian \( H \) such that
\[
\begin{align*}
\sigma(AB) & \geq 0, \\
\mathfrak{N}(\sigma(AB)) & = \mathfrak{N}(H) = \mathfrak{N}.
\end{align*}
\]
if and only if
\[
\begin{align*}
A \mathfrak{N}(H)^\perp & \subseteq \mathfrak{N}(H)^\perp \\
\delta(A \mid \mathfrak{N}(H)^\perp) & = 0.
\end{align*}
\]
If (13) and (14) hold, then
\[
\text{In} H = \text{In}(A \mid \mathfrak{N}(H)^\perp) \leq \text{In} A.
\]
Corollary 1. Let $A$ and $\mathfrak{m}$ satisfy conditions (15) and (16). If $\mathfrak{m}(A H) \geq 0$ and $\mathfrak{m}(H) \supseteq \mathfrak{m}$, then $\mathrm{In} H \subseteq \mathrm{In}(A \mid \mathfrak{m}^+) \subseteq \mathrm{In} A$.

Corollary 2. If $\delta(A) = 0$ and $\mathfrak{m}(A H) \geq 0$, then $\mathrm{In} H \subseteq \mathrm{In} A$. If, in addition, $\delta(H) = 0$ (i.e., $H$ is nonsingular), then $\mathrm{In} H = \mathrm{In} A$.

Corollary 3. If $\mathfrak{m}(A H) \geq 0$ and rank $\mathfrak{m}(A H) = \text{rank} H = \pi(A) + r(A)$, then, again, $\mathrm{In} H = \mathrm{In} A$.

3.3. Suppose the conditions of Theorem 5 are fulfilled and there exists a $K$ such that $\mathfrak{m}(AK) \geq 0$, and $\mathfrak{m}(\mathfrak{m}(AK)) = \mathfrak{m}(K) = \mathfrak{m}$. $A$ and $\mathfrak{m}$ being given. When does every $H$ satisfying $\mathfrak{m}(\mathfrak{m}(AH)) = \mathfrak{m}$ (and not necessarily satisfying $\mathfrak{m}(A H) \geq 0$) also satisfy $\mathfrak{m}(H) = \mathfrak{m}$? For $\mathfrak{m} = (0)$, the question is: When does $\mathfrak{m}(A H) = 0$ imply $H = 0$? The conditions for this are well-known (Corollary below). Thus our Theorem 6 is a generalization of the known Corollary 6.

We require the following definition. If $A$ and $B$ are square matrices (possibly of different orders), we let

$$T(A, B) = \prod_{i,j} (\alpha_i + \beta_j)$$

the product being taken over all pairs of eigenvalues $(\alpha_i, \beta_j)$ of $A$ and $B$, and for the sake of convenience we write $T(A) = T(A, A^*)$. If $A$ is the empty matrix (an operator on a 0-dimensional space), certain consistency conditions force us to take $T(A, B) = 1$.

Theorem 6. Let $\mathfrak{m}$ be a subspace of $V$, and $A$ a matrix for which $A \mathfrak{m}^+ \subseteq \mathfrak{m}^+$. If

\begin{equation}
T(A \mid \mathfrak{m}^+, A^* \mid \mathfrak{m}) \cdot T(A^* \mid \mathfrak{m}) \neq 0
\end{equation}

then $\mathfrak{m}(\mathfrak{m}(AH)) \supseteq \mathfrak{m}$ implies $\mathfrak{m}(H) \supseteq \mathfrak{m}$. Conversely, if

\begin{equation}
T(A \mid \mathfrak{m}^+, A^* \mid \mathfrak{m}) \cdot T(A^* \mid \mathfrak{m}) = 0
\end{equation}

then there exists a Hermitian $H$ such that $\mathfrak{m}(\mathfrak{m}(AH)) = \mathfrak{m}$ but $\mathfrak{m}(H) \supsetneq \mathfrak{m}$.

Corollary 1. There exists a nonzero $H$ such that $\mathfrak{m}(AH) = 0$ if and only if $T(A) = 0$.

Corollary 2. Let $\mathfrak{m}(AH) \geq 0$ and let $\mathfrak{m} = \mathfrak{m}(\mathfrak{m}(AH))$. If $A \mathfrak{m}^+ \subseteq \mathfrak{m}^+$ and (17) holds then $\mathfrak{m}(\mathfrak{m}(AH)) = \mathfrak{m}(H)$.

Corollary 3. Let $\mathfrak{m}(AK) \geq 0$ and $\mathfrak{m} = \mathfrak{m}(K) = \mathfrak{m}(\mathfrak{m}(AK))$. If (17) holds, then $\mathfrak{m}(AH) \geq 0$ and $\mathfrak{m}(\mathfrak{m}(AH)) = \mathfrak{m}$ implies that $\mathfrak{m}(H) = \mathfrak{m}$. Conversely if (18) holds, then there exists a Hermitian $H$ such that $\mathfrak{m}(AH) \geq 0$ and $\mathfrak{m}(\mathfrak{m}(AH)) = \mathfrak{m}$ but $\mathfrak{m}(H)$ is properly contained in $\mathfrak{m}$. 
As in [3], the matrix $A$ is called $H$-stable if, for Hermitian matrices $H$, $AH = (n, 0, 0)$ if and only if $H > 0$. A necessary and sufficient condition for $H$-stability was found in [3], Theorem 4. However, this condition does not greatly facilitate the determination of $H$-stability for a given matrix $A$. Our Theorem 7 below provides an effective test for $H$-stability. The only candidates are nonsingular $A$ with $\mathfrak{R}(A) \geq 0$, and thus we need merely diagonalize $\mathfrak{R}(A)$ and examine the transform of $\mathfrak{g}(A) = (1/2i) (A - A^*)$.

**Theorem 7.** Let $A$ be a nonsingular matrix with $\mathfrak{R}(A) \geq 0$, and let $k = \max_{H \geq 0} \delta(AH)$. Let $S$ be any nonsingular matrix for which $S^*AS = A' = P + iQ$, where $P = P_{11} \oplus 0$ and $Q$ are Hermitian, and $P_{11} > 0$. If $Q$ is partitioned conformably with $P$, then $\text{rank } Q_{22} = k$. In particular, $A$ is $H$-stable if and only if $Q_{22} = 0$.

**Corollary.** If $A$ is an $H$-stable matrix of order $n$, then $\text{rank } \mathfrak{R}(A) \geq n/2$.

**References**


University of Wisconsin