NOTE ON $\Gamma^*$-SEMIGROUPS

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The system $L(S)$ of all nonvoid subsemigroups of a semigroup $S$ is generally a semilattice\(^1\) with respect to the inclusion relation. $L(S)$ is called the subsemigroup semilattice of $S$. In the previous paper [1] we determined all the $\Gamma$-semigroups,\(^2\) i.e., the semigroups whose subsemigroup semilattices are chains. In detail, all the types of $\Gamma$-semigroups are

(1.1) cyclic groups $G(p^n)$ of order of prime power,

(1.2) quasi-cyclic groups $G(p^\infty)$,

(1.3) unipotent semigroups generated by $d$ with each of the following defining relations:

- (1.3.1) $d^2 = d^3$,
- (1.3.2) $d^3 = d^4$,
- (1.3.3) $d^2 = d^{p^n+2}$, $p$ prime,
- (1.3.4) $d^3 = d^{p^n+3}$, $p$ prime $\neq 2$.

In the present note, we shall define $\Gamma^*$-semigroups as generalizations of $\Gamma$-semigroups and shall report the structure of $\Gamma^*$-semigroups except for a part of infinite $\Gamma^*$-groups. The proof will be omitted here but will be given elsewhere.\(^3\)

DEFINITION. A semigroup $S$ is called a $\Gamma^*$-semigroup if every subsemigroup different from $S$ is a $\Gamma$-semigroup.

$S$ is a $\Gamma^*$-semigroup if and only if $L(S)$ is a semilattice satisfying:

Any subset which contains the greatest element is a subsemilattice. A semilattice of this kind is called a $C_0$-semilattice. Obviously all the semigroups of order 2 are $\Gamma^*$-semigroups, and a homomorphic image of a $\Gamma^*$-semigroup is also a $\Gamma^*$-semigroup.

**Lemma 1.** Every element of a $\Gamma^*$-semigroup is of finite order, that is, for any element $x$ there is an idempotent $e$ and a positive integer $n$ such that $x^n = e$.

**Lemma 2.** A $\Gamma^*$-semigroup of order $> 2$ is unipotent. (i.e., an idempotent element is unique).

Generally a unipotent semigroup any element of which is of finite

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1 By a semilattice we mean a partially ordered set in which there is a join of two elements.

2 In [1] we called them $\Gamma$-monoids.

3 *Semigroups and their subsemigroups semilattices*, to appear.

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order is determined by a group and a $Z$-semigroup (i.e., a unipotent semigroup with zero) \[2; 3\]. By Lemmas 1 and 2, we can make the discussion proceed to $\Gamma^*-Z$-semigroups, $\Gamma^*$-groups, and then to the general cases.

**Theorem 1.** Any $\Gamma^*-Z$-semigroup is of order $\leq 4$. All the types of $\Gamma^*-Z$-semigroups other than $\Gamma$-semigroups are listed as follows:

1. \{0, a, b\} of order 3 where $xy = 0$ for all $x, y$,
2. \{0, a, b, c\} of order 4 defined as
   - (2.2.1) $b^2 = c^2 = a$ and other products $= 0$.
   - (2.2.2) $b^2 = cb = c^2 = a$ and other products $= 0$.
   - (2.2.3) $b^2 = c^2 = bc = cb = a$, and other products $= 0$.

As far as the $\Gamma^*$-groups are concerned, we shall limit ourselves to the case of $\Gamma^*$-groups which are properly homomorphic to $\Gamma$-groups.

We can prove that any $\Gamma^*$-group which is properly homomorphic to a $\Gamma$-group has a normal subgroup of index of a prime number. Making use of the theory of finite groups \[4; 5; 6\], we have

**Theorem 2.** Any $\Gamma^*$-group, which is not a $\Gamma$-group and is homomorphic to a $\Gamma$-group of order $> 1$, has one of the following types.

1. The groups of order $pq$ where $p$ and $q$ are different primes. There are two types (3.1).
2. The elementary abelian group: $G(p) \times G(p)$.
3. The generalized quaternion group of order 8.

Incidentally a finite $\Gamma^*$-group, which is not a $\Gamma$-group, is homomorphic to a $\Gamma$-group; a commutative $\Gamma^*$-group which is not a $\Gamma$-group is the direct product of two groups of prime order. Consequently we see that the result of Theorem 2 includes the cases where a $\Gamma^*$-group is homomorphic to a nontrivial finite group or a commutative group. However the problem of determination of the remaining case is still open.

Next, let $S$ be a unipotent $\Gamma^*$-semigroup which is neither a group nor a $Z$-semigroup. Then we can prove that $S$ must be finite. The kernel (i.e., the least ideal) of $S$ is of type $G(p^n)$, and the difference semigroup $D$ of $S$ modulo $G(p^n)$, due to Rees \[7\] is a $Z$-semigroup which has one of the types (1.3.1), (2.1), (2.2.1), (2.2.2), (2.2.3).

Let $e$ be the unique idempotent of $S$, and let $d$ be a generator of $D$. $G(p^{n-1})$ will denote the subgroup of order $p^{n-1}$ of $G(p^n)$.

**Theorem 3.** When $G(p^n)$ is given, we can determine all the unipotent $\Gamma^*$-semigroups, non $\Gamma$-semigroups, whose kernel is $G(p^n)$, by the product of $e$ and $d$ in the following way.
In the case $D$ of order $2$, $S = G(p^n) \cup \{d\}$, $n \neq 0$,

$$ed \in G(p^{n-1}) - G(p^{n-2}).$$

In the case $D$ of order $3$, $D$ is of type (2.1) and $S = G(p^n)$ \cup \{d_1, d_2\}$, $n \neq 0$.

- $ed_1 = ed_2 \in G(p^n) - G(p^{n-1})$,
- $p^n \neq 2$, $ed_1 \neq ed_2$; $ed_1, ed_2 \in G(p^n) - G(p^{n-1})$.

In the case $D$ of order $4$, $S = G(p^n) \cup \{d_1, d_2, d_3\}$, $d_2^2 = d_3^2 = d_1$, $n \neq 0$, $p \neq 2$.

- $D$ of (2.2.1)
- $D$ of (2.2.2) $ed_2 = ed_3 \in G(p^n) - G(p^{n-1})$.
- $D$ of (2.2.3)

According to the above-mentioned theorems, we see that if $S$ is a finite $\Gamma^*$-semigroup, the finite $C_0$-semilattice $L(S)$ satisfies Jordan-Dedekind condition (or $J$-condition cf. [8]). Generally a finite $C_0$-semilattice $K$ satisfying $J$-condition is called a $C_0J$-semilattice. Let $\delta$ denote the dimension of $K$ (cf. [8]), $\lambda$ the breadth, i.e., the number of the maximal chains in $K$, and $\mu$ the order, i.e., the number of elements of $K$.

\[ \begin{align*}
\Gamma\text{-Semigroups} & \quad \text{Idempotent Semigroups of order 2} \\
& \quad \text{(2.1), (3.1)} \quad \text{(2.2)}
\end{align*} \]

**Theorem 4.** A finite $C_0J$-semilattice $K$ is isomorphic to certain $L(S)$ for some finite $\Gamma^*$-semigroup $S$ if and only if $\delta$, $\lambda$, and $\mu$ satisfy the following conditions.

- (5.1) $\delta + \lambda - \mu = 0$,
- (5.2) $\lambda = \alpha + 1$ where $\alpha = 0$ or $1$ or any prime number,

\[ A - B \text{ denotes the set of elements of } A \text{ which are not in } B. \]
\[
\begin{align*}
\text{(3.1), (3.2)} & \\
\text{(3.3)} & \\
\text{(4.1), (4.2), (4.3)} & \\
\end{align*}
\]

\[
\begin{align*}
(5.3) & \quad \begin{cases} 
\text{if } \lambda = 1 \text{ or } 2, \text{ then } \delta \text{ can be taken as an arbitrary positive integer,} \\
\text{if } \lambda = 3, \text{ then } \delta = 2 \text{ or } 3, \\
\text{if } \lambda = p+1, \ p \text{ being a prime number } > 2, \text{ then } \delta = 2. 
\end{cases}
\end{align*}
\]

Finally we shall show the diagrams of \( L(S) \) for a finite \( \Gamma^* \)-semigroup \( S \).

**REFERENCES**