results for operators not assumed positive by means of a reduction procedure [4] and the present theorems.

We are indebted to the work of Eberhard Hopf for suggesting that a resolution of this type is possible.

BIBLIOGRAPHY


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TCHEBYCHEFF QUADRATURE IS POSSIBLE ON THE INFINITE INTERVAL

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The purpose of this announcement is to state a theorem on Tchebycheff quadrature which answers a question posed in [1], and to discuss the proof. Complete details will appear elsewhere.

1. Tchebycheff quadrature.

DEFINITION 1.1. A unit mass distribution on \((-\infty, \infty)\) possessing moments of all positive integer order will be said to belong to class \(D\).

DEFINITION 1.2. Let \(\psi\) be an element of \(D\) and \(n\) a positive integer. We refer to the equations

\[
\frac{1}{n} \sum_{i=1}^{n} x_{i,n}^k = \int x^k d\psi, \quad k = 1, \ldots, n
\]

as the equations \((\psi, n)\). These equations admit a solution \(x_{1,n}, \ldots, x_{n,n}\) which is unique up to permutation of the first index.

DEFINITION 1.3. \(T\) quadrature is said to be possible for an element \(\psi\) of \(D\) if equations \((\psi, n)\) have real solutions for every positive integer \(n\). If \(T\) quadrature is possible for \(\psi\) it is called a \(T\) distribution.

\[\text{---}
\]

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Lemma 1.1 [1]. The mass set of a $T$ distribution lies on a finite interval.

Definition 1.4. $T_1$ quadrature is said to be possible for an element $\psi$ of $D$ if the equations $(\psi, n)$ do not have real solutions for every positive integer $n$, but do have real solutions for an infinite number of positive integers. If $T_1$ quadrature is possible for $\psi$ it is called a $T_1$ distribution. The values of $n$ for which equations $(\psi, n)$ have real solutions are called the $T$ set of $\psi$. If either $T$ or $T_1$ quadrature is possible for $\psi$ in $D$, we say Tchebycheff quadrature is possible.

We are now led to the question raised in [1], namely, is there a $T_1$ distribution whose mass does not lie on a finite interval, or in other words, is Tchebycheff quadrature possible on the infinite interval? Evidence is produced there that this is not so, since it is shown that if a $T_1$ distribution exists whose mass does not lie on a finite interval, its $T$ set would have very large gaps.

Theorem 1.1. There is a $T_1$ distribution whose mass does not lie on a finite interval.

2. Discussion of proof. Lemmas are stated here but not proved. Comments are added which will indicate how the theorem is proved.

Definition 2.1. A simple distribution of degree $n$ is a unit mass distribution consisting of equal masses at $n$ distinct points.

Definition 2.2. Let $\psi, \psi'$ be two elements of $D$, and let $m_k, m'_k$ denote the moments $\int x^k d\psi$, $\int x^k d\psi'$, respectively, $k = 1, \cdots$. Let $n$ be a positive integer. Then $||\psi - \psi'||_n$ is defined as

$$
\max \{ |m_1 - m'_1|, \cdots, |m_n - m'_n| \}.
$$

Lemma 2.1. Let $\psi$ be a simple distribution of degree $n$. There is a number $\epsilon > 0$, called a proximity number of $\psi$, such that if $||\psi - \psi'||_n \leq \epsilon$

where $\psi'$ is any element of $D$, then the equations $(\psi', n)$ have $n$ distinct real solutions.

Lemma 2.2. There is an element $\psi$ of $D$ whose mass is not contained in a finite interval, and an infinite sequence of simple distributions $\psi_k$ of degree $n_k$ and with proximity numbers $\epsilon_k$, $k = 1, \cdots$, where the $n_k$ tend to infinity, such that

$$
(2.1) \quad ||\psi - \psi_k||_{n_k} \leq \epsilon_k, \quad k = 1, \cdots.
$$
COMMENT 1. The condition (2.1) implies that equations $(\psi, n_k)$ have real solutions for $k=1, \cdots$, so that $\psi$ is a $T_1$ distribution.

LEMMA 2.3. Let $\{0_i\}, i=1, \cdots$, be a family of nonoverlapping, finite intervals on the real axis whose union does not lie in a finite interval. There is a sequence of simple distributions $\psi_k$ of degree $n_k$ and proximity numbers $e_k, k=1, \cdots$, where $n_k$ tends to infinity, such that

\[
\int_{0_1} d\psi_{k+p} \geq \gamma_k > 0, \quad k = 1, \cdots, p = 0, \cdots,
\]

and

\[
||\psi_{k+p} - \psi_k||_{n_k} \leq e_k, \quad k = 1, \cdots, p = 1, \cdots.
\]

COMMENT 2. From the $\psi_k$ we can extract a sequence whose limit $\psi$ is in $D$. This distribution $\psi$ and the $\psi_k$ of this lemma satisfy the conditions of Lemma 2.2. The mass of $\psi$ is not on a finite interval because of (2.2), and (2.3) leads to (2.1).

COMMENT 3. In constructing the sequence $\psi_k$ we proceed in a stepwise fashion, constructing $\psi_{k+1}$ from $\psi_k$ in two stages. $\psi_k$ has all its mass on the sets $0_1, \cdots, 0_k$. We move some mass from $0_k$ to $0_{k+1}$, thus creating a mass distribution $\psi'_k$. We then split each mass of $\psi'_k$ into a number of equal masses, locating them close to the mass in which they originated. This can be done so that $\psi_{k+1}$ is simple and has all its mass on $0_1, \cdots, 0_{k+1}$.

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