STABLE HOMEOMORPHISMS CAN BE APPROXIMATED
BY PIECEWISE LINEAR ONES

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A homeomorphism \( h \) of \( E_n \) or \( S_n \) onto itself is stable if \( \exists \) homeomorphisms \( h_1, h_2, \ldots, h_m \) and nonvoid open sets \( U_1, U_2, \ldots, U_m \) such that \( h = h_m h_{m-1} \cdots h_1 \) and \( h_i|_{U_i} = I \) for \( i = 1, 2, \ldots, m \). All orientation preserving homeomorphisms of \( E_n \) or \( S_n \) are stable provided \( n = 1, 2, \) or \( 3 \). There is no example known in any dimension of an orientation preserving homeomorphism which is not stable. In fact, the conjecture that all orientation preserving homeomorphisms of \( E_n \) or \( S_n \) are stable is equivalent to the annulus conjecture (see [3]).

It is known that any homeomorphism of \( E_3 \) onto itself can be approximated by a piecewise linear one (see [2] or [6]). The purpose of this paper is to announce that if \( n \geq 7 \) and \( h \) is a stable homeomorphism of \( E_n \) or \( S_n \) onto itself, then \( h \) can be approximated by a piecewise linear homeomorphism, and also, in the case of \( E_n \), by a diffeomorphism.

The set of all homeomorphisms on \( E_n \) or \( S_n \) forms a group under composition and the subset of stable homeomorphisms forms a normal subgroup. The stable group on \( S_n \) is simple while the stable group on \( E_n \) is not. Due to this fact, there is a shorter proof in the case of \( S_n \) than in the case of \( E_n \), and it is this proof which will be outlined here. The author thanks John Stallings for his assistance.

NOTATION. \( E_n \) is Euclidean \( n \)-space, \( S_{n-1} \) is the unit sphere in \( E_n \), and \( O_n \) is the open unit ball in \( E_n \). Thus \( O_n \cup S_{n-1} = O_n \). For a given integer \( n \), \( O_n \) will usually be denoted by \( O \). If \( U \subseteq E_n \) and \( a > 0 \), \( aU = \{ x \in E_n : \exists y \in U \text{ such that } x = ay \} \). \( C(aU) \), the compliment of \( aU \), will be denoted by \( Ua \). Thus, for a given \( n \), \( aO \) will be the canonical open ball in \( E_n \) of radius \( a \). If \( x, y \in E_n \), \( |x - y| \) will be the usual distance from \( x \) to \( y \). If \( O \) is the origin and \( x \neq O \neq y \), then \( \theta \{ x, y \} \) will represent the angle in radians between the two line intervals, one joining \( O \) to \( x \) and the other joining \( O \) to \( y \). Thus \( 0 \leq \theta \{ x, y \} \leq \pi \). A piecewise linear structure (p.w.l. structure) or combinatorial structure on an open subset of \( E_n \) or \( S_n \) is a triangulation such that the star of each vertex is a combinatorial cell (see §3 of [10]). The identity function will be denoted by \( I \).

The results of this paper are based primarily on Lemma 1 below, a modification of the Engulfing Lemma (see §3.4 of [10]). The proof is omitted.
Lemma 1. Suppose $E_n$ $(n \geq 4)$ has an arbitrary p.w.l. structure $T$, $K$ is a finite subcomplex of $T$, $\dim K \leq n-4$, $a$, $b$, and $e$ are nos. with $0 < a < b$, $e > 0$ and $K \subset bO = bO_n$. Then $\exists$ a homeomorphism $h: E_n \to E_n$ such that $h$ is p.w.l. relative to $T$, $h \mid (a-e)O = I$, $h \mid O(b+e) = I$, $h(aO) \supset K$ and $\theta \{h(x), x\} < \epsilon$ for $x \in E_n$.

The proof of Lemma 2 below follows from Lemma 1 and trivial modifications of §4 of [10] and §8.1 of [11]. The proof is omitted.

Lemma 2. Suppose $E_n$ $(n \geq 7)$ has an arbitrary p.w.l. structure $T$, and $a$, $b$, and $e$ are nos. with $0 < a < b$ and $e > 0$. Then $\exists$ a homeomorphism $h: E_n \to E_n$ such that $h$ is p.w.l. relative to $T$, $h \mid (a-e)O = I$, $h \mid O(b+e) = I$, $h(aO) \supset bO$ and $\theta \{h(x), x\} < \epsilon$ for $x \in E_n$.

Definition. A homeomorphism $h: S_n \to S_n$ is said to have property $P$ if for any p.w.l. structure $T$ on $S_n$ and any $\epsilon > 0$, $\exists$ a homeomorphism $f: S_n \to S_n$ such that $f$ is p.w.l. relative to $T$ and $\theta \{h(x), f(x)\} < \epsilon$ for $x \in S_n$. Let $G_n$ be the set of all homeomorphisms on $S_n$ which possess property $P$.

Observation A. $G_n$ is a normal subgroup of the group of all homeomorphisms under composition.

Proof. The proof that it is a subgroup is immediate. It will be shown that $G_n$ is normal. Suppose $h \in G_n$ and $g: S_n \to S_n$ is any homeomorphism. Show that $g^{-1}hg \in G_n$. Let $T$ and $\epsilon$ be given.

There exists a $\delta > 0$ such that if $\|x-y\| < \delta$, then $\|g^{-1}(x) - g^{-1}(y)\| < \epsilon$. Let $T_1$ be the p.w.l. structure on $S_n$, which is the $g$ image of $T$, $T_1 = g(T)$. Thus if $v$ is a simplex of $S_n$ in the triangulation $T$, then $g(v)$ is a simplex of $S_n$ in the triangulation $T_1$. Since $h \in G_n$, $\exists$ a homeomorphism $f: S_n \to S_n$ which is p.w.l. relative to $T_1$ and with $\theta \{h(x), f(x)\} < \delta$ for $x \in S_n$. Thus $\|g^{-1}hg(x) - g^{-1}fg(x)\| < \epsilon$ for $x \in S_n$. Note that $g^{-1}fg$ is p.w.l. relative to $T$ because: $g$ is p.w.l. from $T$ to $T_1$, $f$ is p.w.l. from $T_1$ to $T_1$ and $g^{-1}$ is p.w.l. from $T_1$ to $T$. This justifies Observation A.

Theorem 1. Let $T$ be an arbitrary p.w.l. structure on $S_n$ $(n \geq 7)$ and let $h: S_n \to S_n$ be a stable homeomorphism. If $\epsilon > 0$, $\exists$ a homeomorphism $f: S_n \to S_n$ such that $f$ is p.w.l. relative to $T$ and $\theta \{h(x), f(x)\} < \epsilon$ for $x \in S_n$.

Proof. The set of all stable homeomorphisms of $S_n$ is a simple, normal subgroup of the group of all homeomorphisms. The fact that it is a normal subgroup is trivial and the fact that it is simple follows from [1] and is even stated explicitly in Theorem 14 of [4]. Therefore, using Observation A, it will follow that $G_n$ contains the stable group if $G_n$ contains some stable homeomorphism distinct from the
identity. This will now be shown.

Let \( h \) be a symmetric radial expansion, i.e., let \( h: E_n \to E_n \) be a homeomorphism such that \( h(x) = x \) for \( ||x|| \geq 1 \), \( h(0) = 0 \), \( \theta \{ h(x), x \} = 0 \) for all \( x \), and if \( 0 < r < 1 \), \( \exists \) a no. \( u(r) \), \( r < u(r) < 1 \) such that \( h[r(\partial - O)] = u(r)(\partial - O) \). Let \( T \) be any p.w.l. structure on \( E_n \) and \( \epsilon > 0 \). It will be shown that \( \exists f: E_n \to E_n \) which is a p.w.l. homeomorphism relative to \( T \) and with \( f(x) = x \) for \( ||x|| \geq 1 \) and \( |h(x) - f(x)| < \epsilon \) for \( x \in E_n \). Since \( h \) determines a homeomorphism from \( S_n \) to itself by defining \( h(\infty) = \infty \), this will show that \( G_n \) is nontrivial and will complete the proof of Theorem 1.

Let \( 0 = r_0 < r_1 < r_2 \cdots < r_{m+1} = 1 \) be nos. such that \( (u(r_{i+2}) - u(r_i)) < \epsilon/2 \) for \( i = 0, 1, 2, \cdots, (m-1) \). By Lemma 3, \( \exists \) p.w.l. homeomorphisms \( f_1, f_2, \cdots, f_m \) such that \( f(x)|_{r_{i+1}O = I, f_i|_{Ou(r_{i+1})} = I, \theta \{ f_i(x), x \} < \epsilon/4 \) for \( x \in E_n \), and \( f_i(r_iO) \supset u(r_i)O \) for \( i = 1, 2, \cdots, m \). Let \( f = f_1 f_2 \cdots f_m \). Now \( f \) is a homeomorphism of \( E_n \) onto \( E_n \) that is p.w.l. relative to \( T \) and \( f|_{C(O)} = I \). It will be shown that \( |f(x) - h(x)| < \epsilon \) for \( x \in O \). Let \( x \in r_{k+1}O \cap Or_k = r_{k+1}O - r_kO, 0 \leq k \leq m \). Then \( f(x) = f_1 f_2 \cdots f_{k+1} \) because \( f_i|r_{k+1}O = I \) for \( i > k + 1 \). In fact, \( f(x) = f_k f_{k+1}(x) \) because \( f_k f_{k+1}(x) \in Ou(r_k) \) and \( f_i|_{Ou(r_k)} = I \) for \( i < k \). (In the special case \( k = 0, f(x) = f_1(x) \).) Now since \( f(x) \) and \( h(x) \in u(r_{k+2})O \cap Ou(r_k), \|f(x)\| \) and \( \|h(x)\| \) differ by \( < \epsilon/2 \). Since \( \theta \{ h(x), f(x) \} < \epsilon/2 \) is measured in radians and any radius under consideration is \( < 1 \), it follows that \( |h(x) - f(x)| < \epsilon \). This completes the proof.

**Theorem 2.** Let \( T \) be an arbitrary p.w.l. structure on \( E_n(n \geq 7) \). If \( h: E_n \to E_n \) is a stable homeomorphism and \( \epsilon(x): E_n \to (0, \infty) \) is a continuous function, then \( \exists \) a homeomorphism \( f: E_n \to E_n \) such that \( f \) is p.w.l. relative to \( T \) and \( |f(x) - h(x)| < \epsilon(x) \) for \( x \in E_n \).

Since the stable group on \( E_n \) is not simple, the trick used in the proof of Theorem 1 cannot be used. A direct construction of \( f \) is required. The proof is omitted.

**Theorem 3.** Suppose \( D \) is any \( C^2 \) differentiable structure on \( E_n(n \geq 7) \). If \( h: E_n \to E_n \) is a stable homeomorphism and \( \epsilon(x): E_n \to E_n \) is a continuous function, then \( \exists \) a homeomorphism \( f: E_n \to E_n \) which is \( C^2 \) diffeomorphism relative to \( D \) and such that \( |f(x) - h(x)| < \epsilon(x) \) for \( x \in E_n \).

**Proof.** Let \( T \) be a \( C^2 \) triangulation of \( E_n \) which is compatible with \( D \) (see [5] or [13]). By Theorem 2, \( h \) may be approximated by a homeomorphism \( f_1 \) which is p.w.l. relative to \( T \). Now by Theorems 5.7 and 6.2 of [7], \( f_1 \) may be approximated by a diffeomorphism \( f \). This completes the proof. The theorem remains true if \( C^2 \) is replaced by \( C^\infty \).
It is not clear whether or not Theorem 3 remains true when \( E_n \) is replaced by \( S_n \). It is known that \( E_n \) and \( S_n \) have to be considered as separate cases. For instance, any two differentiable structures on \( E_n \) are equivalent (except possibly for \( n = 4 \)) while this is not true on \( S_n \) (see [10] and [5] respectively).

Let \( D_1 \) and \( D_2 \) be two differentiable structures on \( E_n \) \((n \geq 7)\). Let \( h: E_n \rightarrow E_n \) be a diffeomorphism mod \( D_1 \). Since diffeomorphisms are always stable, according to Theorem 3, \( h \) can be approximated by \( f \), a diffeomorphism mod \( D_2 \). This type of question might be interesting on \( S_n \). For instance, let \( n = 7 \) and \( D_1 \) be the ordinary differentiable structure on \( S_7 \) and \( D_2 \) be one of Milnor's bad differentiable structures. Can diffeomorphisms mod \( D_1 \) be approximated by diffeomorphisms mod \( D_2 \)? They can be approximated by p.w.l. ones by Theorem 1.

If \( T_1 \) and \( T_2 \) are two p.w.l. structures on \( E_n \) (or \( S_n \)), then any homeomorphism p.w.l. relative to \( T_1 \) can be approximated by one p.w.l. relative to \( T_2 \). This follows from Theorem 2 (resp. Theorem 1) and the fact that p.w.l. homeomorphisms are stable.

References