EXISTENCE THEOREM FOR THE BARGAINING SET $M_i^{(g)}$

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M. Davis and M. Maschler have conjectured [1] that for each coalition structure $B$ in a cooperative game, there exists a payoff vector $x$ such that the payoff configuration $(x; B)$ is stable, i.e., belongs to the bargaining set $M_i^{(g)}$. We outline here a proof of the conjecture. The details of the proof will be published elsewhere.

Let $B = B_1, B_2, \ldots, B_m$ be a fixed coalition structure for an $n$-person game $\Gamma$ with a characteristic function $v(B)$, satisfying $v(B) \geq 0$, and $v(i) = 0$ for $i = 1, 2, \ldots, n$. We denote by $X(B)$ the space of the points $x = (x_1, x_2, \ldots, x_n)$ such that $(x; B)$ is an individually rational payoff configuration (i.r.p.c.). Thus, $X(B) = S_1 \times S_2 \times \cdots \times S_m$, where for $j = 1, 2, \ldots, m$, $S_j$ is the simplex

$$\left\{ x_{B_j} = \{ x_k \}_{k \in B_j} : x_k \geq 0 \text{ and } \sum_{k \in B_j} x_k = v(B_j) \right\} .$$

**Lemma.** Let $c_1(x), c_2(x), \ldots, c_n(x)$ be non-negative continuous real functions defined for $x \in X(B)$. If, for each $x$ in $X(B)$, and for each coalition $B_j$ in $B$, there exists a player $i$ in $B_j$, such that $c_i(x) \geq x_i$, then there exists a point $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ in $X(B)$ such that $c^k(\xi) \geq \xi_k$ for all $k, k = 1, 2, \ldots, n$.

The proof is indirect and one arrives at the contradiction by using Brouwer's fixed point theorem.

Let $(x; B)$ be an i.r.p.c. We shall denote by $(y^B_i, \tilde{x}^{N-B_i}; B)$ an i.r.p.c. which results from the previous one by holding the payments to the players in $N-B_i$ fixed, and giving each player $k$ in $B_j$, $B_j \subseteq B$, an amount $y_k$. Clearly, $x^{N-B_i}$ is the projection of $x$ on the space $S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_m$, and $\tilde{y}^B_i = \{ y_k \}$ is a point in $S_j$.

Let $E_j^i(x)$ be the set of points $\tilde{y}^B_i$ in $S_j$, having the property that in $(y^B_i, \tilde{x}^{N-B_i}; B)$, player $i$, $i \in B_j$, is not weaker than any other player. The set $E_j^i(x)$ is closed and contains the face $y_{i} = 0$ of $S_j$. (See [2].)

We now define for each player $i$, $i = 1, 2, \ldots, n$, the function

$$c_i^k(x) = x_i + \max_{\tilde{y}^B_i \in E_j^i(x)} \min_{k \in B_i} (x_k - y_k).$$

Here, $B_i$ is that coalition of $B$ which contains player $i$.

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1 Throughout this paper we shall use the definitions and the notations of [2].
2 Another proof has been given by the author, M. Davis, and M. Maschler. It has been decided to publish this version, which is simpler.
It can be shown that $c^i(\xi)$ is a non-negative continuous function of $x$.

Since $\sum_{h \in B_j} x_h = \sum_{h \in B_j} y_h = v(B_j)$, it follows that $c^i(x) \leq x_i$ for all $i$, $i=1, 2, \ldots, n$. Let $E_i$, $i=1, 2, \ldots, n$, be the set of points $x$, $x \in X(B)$, for which $i$ is not weaker than any other player of the coalition $B_j$ which contains player $i$. Clearly, $(x; B) \in M_i^0$ if and only if $x \in \bigcap_{k=1}^n E_k$. If $x \in E_i$, then its projection $\hat{x}^{B_i}$ on $S_i$ belongs to $E_j(x)$. In this case $c^i(x) = x_i$. Conversely, if $c^i(x) = x_i$, then some $y^{B_i} \in E_j(x)$ must be equal coordinatewise to $x^{B_i}$, hence $x \in E_i$.

It is proved in [2] (see proof of Theorem 2), that for each $x$, $x \in X(B)$, and for each coalition $B_j$, $B_j \in B$, there exists a player $i$, $i \in B_j$, such that $x \in E_i$. Thus, for this player, $c^i(x) = x_i$. By the lemma, there exists a point $\xi$, $\xi \in X(B)$, such that $c^k(\xi) = \xi_k$ for all $k$, $k = 1, 2, \ldots, n$. Therefore, $\xi \in \bigcap_{k=1}^n E_k$, and so $(\xi, B) \in M_i^0$. We have thus proved:

**Theorem.** Let $B$ be a coalition structure in an $n$-person cooperative game; then there always exists a payoff vector $x$ such that $(x; B) \in M_i^0$.

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**References**


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