RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

CONVOLUTION OF SEQUENCES

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A summability method is a linear functional on a space of sequences S. The method \( \theta \) is said to be regular if, for each convergent sequence \( s = \{ s_n \} \) we have \( \theta(s) = \lim_{n \to \infty} s_n \). In this paper we define various types of convolution (multiplication) of sequences; we use the symbol \( * \) to denote convolution. Our convolution is always distributive, but not necessarily associative or commutative. We consider the regular methods \( \phi \) such that \( \phi(s * t) = \phi(s) \phi(t) \) for all sequences \( s \) and \( t \) in the domain of \( \phi \), \( S(\phi) \), that is, the regular homomorphisms from \( S(\phi) \) to the real numbers. We write \( s(\phi) = \phi(s) \) for each sequence \( s \) in \( S(\phi) \) and we impose the weak topology on the set of homomorphic methods. In case the multiplication is commutative and associative and we were dealing with complex sequences, then \( s(\phi) \) would be a complex Banach algebra, \( s(\phi) \) would be the Fourier transform of the sequence \( s \), and the weak topology on the set of homomorphic methods would yield the maximal ideal space of \( s(\phi) \). Although we shall deal with real sequences, we shall use a certain amount of Gel'fand theory.

The types of convolution to be considered are:

(a) Pointwise multiplication—if \( s \) and \( t \) are two bounded sequences then \( s * t = \{ s_n t_n \} \).

(b) Cauchy multiplication—if \( s \) and \( t \) are two sequences such that

\[
S(z) = \sum_{n=0}^{\infty} a_n z^n, \quad T(z) = \sum_{n=0}^{\infty} b_n z^n, \quad (a_n = s_{n+1} - s_n, \quad b_n = t_{n+1} - t_n)
\]

are analytic and bounded in the unit circle \( D \) in the complex \( z \)-plane, then \( s * t = \{ \sum_{k=0}^{n} \sum_{j=0}^{k} a_j b_{k-j} \} \). We note that the power corresponding to \( s * t \) is \( S(z) T(z) \).

(c) If \( s \) and \( t \) are bounded sequences, and \( B = (b_{nk}) \) is a positive
regular triangular summation matrix, we define $s \ast t = \{ \sum_{k=0}^{n} b_{nk}s_k t_k \}$.

(d) If $s$ and $t$ are two bounded sequences, $B$ has all properties stated in (c), and, in addition,
\[ \lim b_{n,n-r} = 0, \quad r = 0, 1, \ldots, \]
then
\[ s \ast t = \{ \sum_{k=0}^{n} b_{nk}s_k t_{n-k} \}. \]

Convolutions (a) and (b) are commutative and associative, convolution (c) is commutative but not associative while convolution (d) is neither commutative nor associative. If $\phi$ is a regular homomorphism relative to (a), (c), or (d) we turn $S(\phi)$ into a Banach space by imposing the norm $\|s\| = \sup |s_n|$; if $\phi$ is a homomorphism relative to (b) we use the norm $\|s\| = \sup S(z)$, the supremum being taken over all points $z$ in $D$.

**THEOREM 1A.** If $\phi$ is a regular homomorphism relative to (a) or (c), and $s$ is in $S(\phi)$, then $\phi(s)$ is a cluster value of $s$.

We first show that $\lim \inf s \leq \phi(s) \leq \lim \sup s$. If $\phi(s) = \sigma > \lim \sup s$ then $\lim \sup [s/\sigma]^{(m)} \to 0$ as $m \to \infty$ (here $s^{(m)}$ denotes the sequence $s$ convolved with itself $m$ times). We use the fact that $\phi$ is a linear continuous functional to conclude that $\phi(s/\sigma)^{(m)} \to 0$. Since $\phi$ is a homomorphism, we must have $\phi(s/\sigma)^{(m)} = 1$ for all $m$. We have a contradiction; thus $\phi(s) \leq \lim \sup s$. Similarly we see that $\phi(s) \geq \lim \inf s$. In particular $\phi$ must evaluate the sequence $(s - \sigma)^{(2)}$ to 0, since $(s - \sigma)^{(2)}$ is a non-negative sequence when the convolution considered is (a) or (c), 0 must be a cluster value of $s - \sigma$. In other words, $\sigma$ must be a cluster value of $\phi(s)$.

**THEOREM 1B.** If $\phi$ is a regular homomorphism relative to (b), and $s \in S(\phi)$ satisfies
\begin{equation}
\sup |s_n| \leq M \sup_{z \in D} |S(z)|
\end{equation}
for some constant $M$, then $\phi(s)$ is a cluster value of $S(z)$ as $z \to 1$.

**THEOREM 1C.** If $\phi$ is a regular homomorphism relative to (d), then $\lim \inf s \leq \phi(s) \leq \lim \sup s_n$, for each sequence $s$ in $S(\phi)$.

When $\phi$ is a homomorphism relative to (d), we cannot imitate the proof of Theorem 1A to conclude that $\phi(s)$ must be a cluster value of $s$; with this convolution $s \ast s$ may be negative.

**THEOREM 2A.** Suppose that the sequence $s$ satisfies (1). If $s$ is Abel
summable and evaluated by some method \( \phi \) which is a regular homomorphism relative to (b), then \( \phi(s) \) must equal the Abel sum of \( s \).

**Theorem 2B.** If the bounded sequence \( s \) is evaluated to \( \sigma \) by the matrix \( B \) and it is in \( S(\phi) \), where \( \phi \) is a regular homomorphism relative to (c) or (d), then \( \phi(s) = \sigma \).

Theorem 2A follows from Theorem 1B; to prove Theorem 2B we note that if the matrix \( B \) evaluates \( s \) to \( \sigma \), then \( s \ast 1 \rightarrow \sigma \).

**Theorem 3A.** If \( \phi \) is a regular homomorphism relative to convolution (a) or (c), and \( s \) is a sequence in \( S(\phi) \) which is bounded away from 0, then \( \{1/s_n\} \) is in \( S(\phi) \); if \( s \) and \( t \) are in \( S(\phi) \), then the sequences \( s \lor t = \max(s_n, t_n) \) and \( s \land t = \min(s_n, t_n) \) are in \( S(\phi) \).

If \( \phi \) is a regular homomorphism relative to (a) or (c), \( s \in S(\phi) \), and \( \phi(s) = \sigma \), then \( (s - \sigma)^{(2)} \) is a non-negative sequence which \( \phi \) evaluates to 0. Consequently, if \( \epsilon \) is a positive number, the set of integers \( n \), on which \( |s_n - \sigma| > \epsilon \) is sparse. The same must be true for the set of integers on which \( |1/s_n - 1/\sigma| > \epsilon \) and \( \phi(\{1/s_n\}) = 1/\sigma \).

To show that \( s \lor t \) is in \( S(\phi) \), we note that if \( \phi(s) = \sigma, \phi(t) = \tau \), then there exist subsequences \( \{s_{n_j}\}, \{t_{m_j}\} \) which converge to \( \sigma \) and \( \tau \); moreover the sequences of integers \( \{n_j\} \) and \( \{m_j\} \) are fairly dense. Consequently, the sequence \( \{n_j\} \cap \{m_j\} \) is also fairly dense and \( s \lor t \) has a subsequence converging to \( \max(\sigma, \tau) \) along this intersection. Hence \( \phi(s \lor t) = \max(\phi(s), \phi(t)) \), and similarly \( \phi(s \land t) = \min(\phi(s), \phi(t)) \).

This theorem could have been proved by Banach algebra theory in the case where \( \phi \) is a regular homomorphism relative to (a). By such a method we can prove:

**Theorem 3B.** If \( \phi \) is a homomorphism relative to (b), and \( s \) is a sequence in \( S(\phi) \) such that the corresponding power series \( S(z) \) is bounded away from 0 and (1) is satisfied, then the sequence corresponding to \( 1/S(s) \) is in \( S(\phi) \).

**Theorem 4.** If \( \phi \) is a regular homomorphism relative to (b), and \( \{s_n\} \) is a sequence in \( S(\phi) \), then \( \{s_{n+1}\} \) is in \( S(\phi) \) and \( \phi(\{s_{n+1}\}) = \phi(s_n) \).

Let \( \phi_0 \) be a regular homomorphism and let \( \phi \) denote the set of all homomorphisms \( \phi \) such that \( S(\phi) \supseteq S(\phi_0) \). According to the weak topology a regular homomorphism is in the closure of a set \( \{\phi_n\} \) if and only if \( s(\phi_0) \) is a cluster value of the set \( \{s(\phi_n)\} \) for each \( s \) in the common convergence field. We denote the topological spaces formed by \( \Phi_a, \Phi_b, \Phi_c, \Phi_d \), according as the convolution is (a), (b), (c) or (d).
THEOREM 5A. The spaces $\Phi_d$ and $\Phi_e$ are totally disconnected.

The proof depends on the fact that if $\phi$ is a regular homomorphism relative to (a) or (c), then each sequence in $\mathcal{S}(\phi)$ has a very dense subsequence which converges to $\phi(s)$.

Now suppose that $s$ is a sequence such that the corresponding power series $S(z)$ is analytic in $|z| < 1$. If $\sigma$ is a number between $\lim sup_{z \to z_n} S(z)$ and $\lim inf_{z \to z_n} S(z)$, then there exists a sequence of points $\{z_n\}$ such that $z_n \to 1^-$ and $S(z_n) \to \sigma$. The functional $\phi(s)$ $= \lim_{z_n \to 1^-} S(z_n)$ is a regular homomorphism relative to (b). In other words, for regular homomorphisms relative to (b), $\mathcal{S}(\phi)$ takes on each value between its upper and lower bound. Thus

THEOREM 5B. The space $\Phi_b$ contains a continuum.

The following is an example of a totally disconnected space $\Phi_d$. Let the matrix $B = (b_{nk})$, defining the convolution, be given by

$$
b_{n,n/2} = 1, \quad b_{nk} = 0, \quad k \neq n/2, \quad n \text{ even},$$

$$b_{n,k} = 1/(n+1), \quad k \leq n, \quad b_{n,k} = 0, \quad k > n, \quad n \text{ odd}.
$$

Let the method $\phi_0$ be defined by the matrix $A = (a_{nk})$ where

$$a_{n,n} = 1, \quad a_{n,k} = 0, \quad k \neq n, \quad n \text{ even},$$

$$a_{n,n-1} = 1, \quad a_{n,k} = 0, \quad k \neq n - 1, \quad n \text{ odd}.
$$

The set of regular homomorphisms $\phi$ such that $\mathcal{S}(\phi) \supseteq \mathcal{S}(A) = \mathcal{S}(\phi_0)$ forms a totally disconnected space $\Phi_d$ under our weak topology.

I do not know whether spaces $\Phi_d$ containing a continuum exist.

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