EXAMPLE OF A PROPER SUBGROUP OF $S_\omega$ WHICH HAS A SET-TRANSITIVITY PROPERTY

BY GERALD STOLLER
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S. M. Ulam, on page 33 of his book, *A collection of mathematical problems*, poses the following question: Let $G$ be a subgroup of $S_\omega$ [the group of all permutations of the integers] with the property that for every two sets of integers of the same power whose complements are also of the same power, there exists a permutation $g$ of $G$ which transforms one set into the other. Is $G = S_\omega$ (Chevalley, von Neumann, et al.)?

The answer to this question is no!

We change the problem immaterially by taking $S_\omega$ to be the group of all permutations of the natural numbers rather than the integers; this is helpful since all infinite subsets of the natural numbers are order-isomorphic. A subgroup $G$ which is transitive (wherever possible) on the set of all subsets of the natural numbers is defined by means of a finiteness condition.

Let $N$ be the set of natural numbers with the usual ordering. Consider the set $G$ of all $\sigma \in S_\omega$ satisfying the condition:

(F) there exist $A_1, A_2, \cdots, A_k, B_1, B_2, \cdots, B_k$ subsets of $N$ such that $\bigcup_{i=1}^k A_i = N = \bigcup_{i=1}^k B_i$ and in addition, for all $i$, $\sigma: A_i \mapsto B_i$ is an order-isomorphism.

Call \{$(A_1, B_1), (A_2, B_2), \cdots, (A_k, B_k)$\} a class of order-pairs for $\sigma$.

Let $\sigma, \tau \in G$ where \{$(A_1, B_1), \cdots, (A_k, B_k)$\} & \{$(C_1, D_1), \cdots, (C_q, D_q)$\} are classes of order-pairs for $\sigma$ & $\tau$ respectively. It is easily seen that \{$(\sigma^{-1}[B_i \cap C_j], \tau[B_i \cap C_j]): i \in \{1, \cdots, k\} & j \in \{1, \cdots, q\}$\} is a class of order-pairs for $\sigma \tau$, so that $\sigma \tau \in G$. Also $\sigma^{-1} \in G$ since \{$(B_1, A_1), (B_2, A_2), \cdots, (B_k, A_k)$\} is a class of order-pairs for $\sigma^{-1}$. Consequently (since $G$ is obviously nonempty) $G$ is a subgroup of $S_\omega$.

That $G$ has the property stated in the problem is clear since subsets of $N$ having the same power are order-isomorphic. The (at most) two order-isomorphisms needed allow us to define an element of $G$ as required.

An element $\rho$ of $S_\omega$ which reverses arbitrarily long strings of natural numbers cannot be in $G$. For example, $\rho$ can be given by: $\rho(m) = (n+1)^2 - (m + 1 - n^2)$ where $n^2 \leq m < (n+1)^2$. Suppose that $\rho$ satis-

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fies (F) with \((A_1, B_1), \cdots, (A_k, B_k)\) as a class of order-pairs. The
\(2k+1\) integers \(k^2, \cdots, k^2+2k\) are reversed by \(\rho\), but two of them
must fall in the same set \(A_i\). This is a contradiction.

Therefore \(G\) is a proper subgroup of \(S_\omega\).

Harvard University

ON THE ISOMORPHISM PROBLEM FOR BERNOULLI SCHEMES

BY J. R. BLUM\(^1\) AND D. L. HANSON\(^2\)

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1. DEFINITION 1. A Bernoulli scheme \((E, \Omega, \mathcal{F}, P, T)\) is a probabil-

ity space together with a transformation \(T\), where

(i) \(E = \{1, \cdots, n\}\) for some positive integer \(n\), or \(E = \{1, 2, \cdots\}\),

(ii) \(\Omega = \{\omega = (\cdots, \omega_{-1}, \omega_0, \omega_1, \cdots) \mid \omega_i \in E \text{ for all } i\}\),

(iii) \(\mathcal{F}\) is the smallest \(\sigma\)-algebra containing all sets \(A^*_i = \{\omega \mid \omega_i = k\}\),

(iv) \(q_k > 0\) is defined for \(k \in E\) with \(\sum_{k \in E} q_k = 1\), \(P\) is the product

measure on \(\mathcal{F}\) defined by \(P\{A^*_i\} = q_k\) for all \(i\),

(v) \(T\) is the shift transformation defined on \(\Omega\), i.e., \(T\omega = \omega'\) if

and only if \(\omega_i' = \omega_{i+1}\) for all \(i\).

We shall sometimes refer to a Bernoulli scheme as a \((q_1, \cdots, q_n)\)-
scheme or a \((q_1, q_2, \cdots)\)-scheme depending upon whether
\(E = \{1, \cdots, n\}\) or \(E = \{1, 2, \cdots\}\).

DEFINITION 2. Two Bernoulli schemes \((E, \Omega, \mathcal{F}, P, T)\) and
\((E', \Omega', \mathcal{F}', P', T')\) are said to be isomorphic modulo sets of measure
zero (or simply isomorphic) if there exist sets \(D \subseteq \mathcal{F}, D' \subseteq \mathcal{F}'\) and a map-
ing \(\phi: D \to D'\) such that

(i) \(TD = D\),

(ii) \(\phi: D \to D'\) is one-to-one and onto,

(iii) \(\phi(T\omega) = T'(\phi\omega)\) for all \(\omega \in D\),

(iv) if \(A \subseteq D\) then \(A \subseteq \mathcal{F}\) if and only if \(\phi A \subseteq \mathcal{F}'\),

(v) if \(A \subseteq D\) and \(A \subseteq \mathcal{F}\) then \(P(A) = P'(\phi A)\),

(vi) \(P(D) = 1\).

DEFINITION 3. The entropy of a \((q_1, \cdots, q_n)\)-scheme \([\{q_1, q_2, \cdots\}]\)-
scheme is given by

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