DIFFERENTIAL INEQUALITIES

BY RAYMOND M. REDHEFFER

Communicated by Lipman Bers, August 14, 1962

Let $B$ denote a bounded region in Euclidean $n$-space, with boundary $\partial B$ and closure $\overline{B}$. We write $P = x = (x_1, x_2, \ldots, x_n) \in \overline{B}$, $u_i = \partial u/\partial x_i$, $u_{ij} = \partial^2 u/\partial x_i \partial x_j$, and similarly for $v$, $c$ and $y$. The normal derivative $u_\nu$ is understood in the sense of Walter, namely:

$$u_\nu(P_0) = \limsup [u(P_k) - u(P_0)] |P_k - P_0|^{-1}$$

where $P_k \subseteq B$, $P_0 \subseteq \partial B$, and $P_k \to P_0$ in such a way that

$$(P_k - P_0) \cdot P_0$$

tends to a fixed vector, $v$. We have $u = u(x)$, $v = v(x)$, and Independent variables are denoted by the letter $s$. The letter $\epsilon$ means "+" or "−," and has the same meaning in hypothesis and conclusion. We suppose $\epsilon \neq 0$ and $\delta \neq 0$ to be nonnegative constants. The statement "$f(x, v, v_i, v_{ij})$ is monotone" means that

$$\epsilon |f(x, v, v_i, v_{ij}) - f(x, v, v_0, s_{ij})| \geq 0$$

when the matrix $\epsilon [v_{ij} - (s_{ij})]$ is nonnegative. Other assertions of monotony are interpreted similarly. We assume $u \in C^{(2)}$, $v \in C^{(2)}$ in $B$ and $u \in C$, $v \in C$ in $\overline{B}$, although discontinuities can be allowed as in [2].

It is convenient to write $v' = (v, v_i, v_{ij})$, a vector of $1 + n + n^2$ components, and similarly for $u$, $s$, and $y$. Also $f' = (f_u, f_{ui}, f_{uij})$ with argument $(x, v')$ or $(x, s')$, as the case may be. Similarly, $k' = (k_u, k_{u})$. The statement "$f'$ is continuous in the neighborhood of $v'$" means that there is an $h > 0$ such that $f'(x, s')$ is continuous for $|s' - v'| < h$. Other statements of this kind are understood similarly.

THEOREM I. Let $k(x, u \downarrow, u_\nu)$ be strictly monotone, let $k(x, v, v_\uparrow)$ be monotone, and let $f'$ be continuous in the neighborhood of $v$. Suppose further:

(i) $f(x, u \downarrow, u_i, u_{ij})$ is monotone, and $f(x, s, s_i, s_{ij} \uparrow)$ is monotone in the neighborhood of $v$.

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(ii) To every compact subset $S \subset B$ corresponds a function $c(x) \in C^2$ such that

$$\sum f_{ij}(x, v') c_i + \sum f_{ij}(x, v') c_{ij} > 0$$

at those points of $S$ (if there are any) at which

$$f_v(x, v') = \sum f_{ij}(x, v') c_i c_{ij} = 0.$$

Then $p(Tu - Tv) \leq 0$ and $p(Ru - Rw) \leq 0 \Rightarrow p(u - v) \leq 0.$

To prove the theorem let $y \in C$, $y_0 \in C$, and suppose the conclusion violated. For small $h > 0$ the function $w = p(u - v) - hy$ has an interior maximum and at that point $w_0 = p(u - v) - hy_0 > 0$. We choose $y = -\mu \{c(x)\}$ for a suitable function $\mu$, and $y_0 = -f_v(x, v')$.

Let $2c(x) = r^2 r_0 - r_0$ where $r$ is the distance to a fixed point $P_0$ and where $r_0$ is constant. The unit normal, $v$, to the sphere $r = |r_0|$ is $v = e_i$. A point $P$ is called a sphere-point $(r_0, v)$ of the set $u = v$ if $P$ is on the sphere $r = |r_0|$, if $u = v$ at $P$, and if there is a neighborhood $N$ of $P$ such that $p(u - v) < 0$ in those points of $N$ at which $c(x) > 0$.

Thus, when $r_0 > 0$ the set $u = v$ lies locally inside a sphere of radius $r_0$ and outer normal $v$, whereas if $r_0 < 0$ the set lies locally outside a sphere of radius $|r_0|$ and inner normal $v$. The following result affords a smooth transition from the weak to the strong maximum principle:

**Theorem II.** Let $f'$ be continuous in the neighborhood of $v$, let $f(x, s, s_i, s_{ij})$ be monotone in the neighborhood of $v$, and suppose further:

(i) At the point $P \in B$, either

$$\sum v_{ij} f_{ij}(x, v') + r_0^{-1} \sum f_{ij}(x, v') > 0$$

or

$$\sum f_{ij}(x, v') v_i v_j > 0.$$

(ii) In a neighborhood of $P$, $p(Tu - Tv) \leq 0$.

Conclusion: $P$ is not a sphere-point $(r_0, v)$ of the set $u = v$.

The Fréchet derivative is

$$\lim_{h \to 0} (T(s + hy) - T(s)) h^{-1} = -f'(x, s) \cdot y' = L(s)y$$

where $L(s)$ is, for each $s$, a linear operator on $y$. Similarly,

$$\lim_{h \to 0} (R(s + hy) - R(s)) h^{-1} = M(s)y$$

where $M(s)$ is linear. We say that the pair of operators $(L, M)$ be-
longs to the class \((E, D, A)\) if \(E, D, A\) are positive constants such that the problem

\[
Ly \geq E, \quad x \in B; \quad My \geq D, \quad x \in \partial B
\]

has a solution \(y \in C^{(2)}, \quad 0 \leq y \leq 1, \quad \|y_i\| + \|y_{ij}\| \leq A.\)

**Theorem III.** Let \(f'(x, s')\) and \(k'(x, s', s_i)\) be uniformly equicontinuous in \(s\) and let \(\sup |f_v(x, v')| < \infty, \sup |k_v(x, v, v_i)| < \infty.\) Suppose further for all \(s:\)

(i) The matrix \([f_{ij}(x, u, u_i, u_{ij})]\) \(\geq 0,\) and \(k_{iv}(x, u, s_i) \geq 0.\)

(ii) \([L(s), M(s)] \in (E, D, A).\)

Conclusion: \(p(Tu - Tv) \leq \varepsilon_p\) and \(p(Ru - Rv) \leq \delta_p \Rightarrow p(u - v) \leq \max ((\varepsilon_p/E), (\delta_p/D)).\)

The proof follows by constructing a suitable family of solutions \(y(x, \xi)\) of

\[
p[T(v + py) - Tv] > \varepsilon_p, \quad p[R(v + py) - Rv] > \delta_p,
\]

and using the fundamental theorem of Nagumo [3].

Let \(c(x) \in C^{(2)}\) be a fixed function with \(\inf c(x) = 0, \sup \|c_i(x)\| = 1.\) The constants \(C = \sup c(x), \quad C_2 = \sup \|c_{ij}(x)\|\) measure the size of \(B\) with respect to \(c.\) The function

\[
U(p, \beta) = \inf p[f(x, u, u_i, u_{ij} + \rho c_i + \rho \beta c_{ij}) - f(x, u, u_i, u_{ij})]
\]

for \(\alpha \geq 0, \beta \geq 0\) measures the influence of the second-derivative terms in \(f.\) We write \(V\) instead of \(U\) when \(v(x)\) instead of \(u(x)\) occurs on the right. The influence of the first-derivative terms is expressed by

\[
p[f(x, u, u_i, s_{ij}) - f(x, v, v_i, s_{ij})] \leq G_p(S_2, \|u_i - v_i\|) \quad \text{for} \quad p(u - v) > 0
\]

where \(S_2 = \sup \|s_{ij}\|,\) and where \(G_p\) is continuous and monotone in both arguments. For simplicity let

\[
Ru = u - k(x, u), \quad k(x, v_i + s) - k(x, v_i) \leq \gamma(\|s\|)
\]

where \(\gamma\) is continuous and increasing. Under these conditions we have:

**Theorem IV.** Let \(f(x, u, u_i, s_{ij})\) and \(k(x, u, s_i)\) be monotone and suppose that \(\eta(s),\) for \(0 < s < C,\) is a positive nondecreasing solution of the differential inequality

\[
U_p(\eta, \eta') > \varepsilon_p + G_p(V_2, \eta), \quad V_2 = \sup \|v_{ij}\|,
\]

or of the inequality

\[
V^{-p}(\eta, \eta') > \varepsilon_p + G_p(V_2 + \eta' + \eta C_2, \eta).
\]

Then \(p(Tu - Tv) \leq \varepsilon_p\) and \(p(Ru - Rv) \leq \delta_p\) implies
\[ p(u - v) \leq \delta^p + \gamma[\eta(C)] + \int_{c(x)}^C \eta(s) \, ds. \]

The proof follows by setting \( \mu'(s) = \eta(s) \), \( \gamma = m - \mu[c(x)] \), where \( m \) is a constant so chosen that the function \( p(u - v) - \gamma \) does not assume a positive maximum on \( \partial B \).

BIBLIOGRAPHY


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COMPLETE LOCALLY AFFINE SPACES AND ALGEBRAIC HULLS OF MATRIX GROUPS

BY LOUIS AUSLANDER

Communicated by I. M. Singer, November 27, 1962

Let \( M \) be a complete Riemann manifold with curvature and torsion zero. If \( \pi_1(M) \) denotes the fundamental group of \( M \), then Bieberbach [3; 4] proved that \( \pi_1(M) \) contains an abelian normal subgroup of finite index. Moreover, if \( M \) is compact then \( M \) is covered by a torus.

In recent years the study of general affine connections has led to the study of the following problem: How can one classify the manifolds which possess a complete affine connection with curvature and torsion zero? Such manifolds will be called complete locally affine spaces.

It was Zassenhaus [6] who first gave a general setting to the Bieberbach theorem. He showed a special case of the following theorem:

**Theorem 1.** Let \( G \) be a connected Lie group with its radical \( R \) simply connected, \( \rho: G \to G/R \) the projection, and \( L \) a closed subgroup of \( G \). If the identity component \( L_0 \) of \( L \) is solvable, then the identity component of the closure of \( \pi_1(L) \) is solvable.

This theorem in this generality is due to H. C. Wang [5] and his

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1 With partial support from the N. S. F.