A TOPOLOGICAL CLASSIFICATION OF CERTAIN 3-MANIFOLDS

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Communicated by Deane Montgomery, December 12, 1962

Introduction. In [1] J. Stallings proves that members of the class of closed irreducible 3-manifolds which are fibered over a circle by an aspherical 2-manifold may be distinguished from other closed irreducible 3-manifolds by their fundamental group alone.

He asks whether two members of this class of 3-manifolds are homeomorphic if they have isomorphic fundamental groups. This question is answered in the affirmative here, thus giving a classification of these manifolds according to their fundamental group.

The closed case. Let us denote by \( \mathcal{M} \) the class of all 3-manifolds satisfying the following conditions:

(a) Manifolds of \( \mathcal{M} \) are irreducible (every 2-sphere bounds a 3-cell).

(b) Manifolds of \( \mathcal{M} \) are closed.

(c) Manifolds of \( \mathcal{M} \) have fundamental groups which contain a finitely generated normal subgroup of order \( >2 \), with quotient group an infinite cyclic group.

Theorem 1. Let \( M_2 \) be any closed irreducible 3-manifold. Let \( M_1 \) belong to \( \mathcal{M} \), then \( M_1 \) is homeomorphic to \( M_2 \) if and only if \( \pi_1(M_1) \) is isomorphic to \( \pi_1(M_2) \).

Proof. One direction is trivial. By Stallings' theorem [1] \( M_1 \) admits a fibering over \( S^1 \), with fiber a closed 2-manifold \( T_1 \). Let

\[
1 \rightarrow H_1 \rightarrow G_1 \rightarrow Z \rightarrow 0
\]

denote the sequence of fundamental groups of \( T_1, M_1, S^1 \), respectively corresponding to this fibering. Let \( \rho^* \) denote the assumed isomorphism from \( \pi_1(M_1) = G_1 \) to \( \pi_1(M_2) = G_2 \). Then \( \rho^* \) induces

\[
1 \rightarrow H_2 \rightarrow G_2 \rightarrow Z \rightarrow 0.
\]

Now, \( G_1 \) and \( G_2 \) are both described by giving the automorphisms \( \phi^*_1, \phi^*_2 \) of \( H_1, H_2 \), which are induced by a generator of \( Z \), pulled back to \( G_1, G_2 \), and then acting on \( H_1, H_2 \) by conjugation.

Since \( \rho^* \) is an isomorphism we may assume

\[
\rho^* \phi^*_1 = \phi^*_2 (\rho^* | H_1). \tag{3}
\]

372
According to Stallings theorem [1] there is a fibering of $M_2$ which induces (2). Denote by $T_2$ the fiber of this map. Cut $M_1$, $M_2$, along a fiber, obtaining $T_1 \times I$, $T_2 \times I$. Denote by $\phi_i: T_i \times 0 \to T_i \times 1$ the maps which repair these cuts. Clearly $\phi_i$ induces $\phi_i^*$ modulo an inner automorphism of $H_i$.

Now if a homeomorphism $\rho: T_1 \times I \to T_2 \times I$ can be found satisfying
\[(4) \quad \phi_2(\rho| T_1 \times 0) = \rho \phi_1,\]
then $\rho$ defines a homeomorphism from $M_1$ to $M_2$.

An algebraic map $\rho^*$ from $\pi_1(T_1)$ to $\pi_1(T_2)$ is already defined, so according to a theorem of Nielson [2], and Mangler [4], there exists a homeomorphism $\rho_1: T_1 \to T_2$ such that $\rho_1^* = \rho^*$. Now\(^1\) $(\rho_1 \phi_1)^* = (\phi_2(\rho_1| T_1 \times 0))^*$. According to a theorem of Baer [3] (for orientable surfaces) and Mangler [4] (for orientable and nonorientable surfaces), the maps $\rho_1 \phi_1$ and $\phi_2(\rho_1| T_1 \times 0)$ differ by an isotopy of $T_2$. Let us call this isotopy $h_t$. Then $h_0 \circ \rho_1 \circ \phi_1 = \rho_1 \circ \phi_1, \quad h_1 \circ \rho_1 \circ \phi_1 = \phi_2(\rho_1| T_1 \times 0)$. Define $\rho: T_1 \times I \to T_2 \times I$ as follows:
\[\rho(x, t) = (ht\rho_1, t)\]
then
\[\rho(x, 1) = (h_1\rho_1, 1)\]
\[\rho(x, 0) = (h_0\rho_1, 0) = (\rho_1, 0).\]

So that
\[\rho \phi_1 = h_1\rho_1 \phi_1 = \phi_2(\rho_1| T_1 \times 0)\]
but
\[\phi_2(\rho| T_1 \times 0) = \phi_2(h_0\rho_1| T_1 \times 0)\]
\[= \phi_2(\rho_1| T_1 \times 0)\]
and so (4) is satisfied and the theorem is proved.

**The compact case.** As far as the compact nonclosed case is concerned a somewhat different approach may be adopted.

Suppose $M_1$ is a compact, orientable, irreducible, 3-manifold, and $\pi_1(M_1) = G_1$ contains a normal subgroup $H_1$ such that:
(a) $H_1$ is finitely generated.
(b) $G_1/H_1 \approx \mathbb{Z}$.
(c) $H_1 \approx \mathbb{Z}_2$.\(^2\)

Having already investigated the case $\partial M_1 = \phi$, we may assume $\partial M_1 \neq \phi$. According to Stallings theorem [1] $M_1$ is fibered over $S^1$ with fiber a 2-manifold $S_1$. Since $M_1$ is orientable this fibering implies

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\(^1\) Modulo an inner automorphism.
\(^2\) This constitutes part of the hypothesis of Theorem 2.
each boundary component of $M_1$ is a torus. Denote by $T_1, \ldots, T_n$ these boundary tori. Since each boundary torus has a fibering induced in it, we may select curves $m_i, l_i$ in each $T_i$ such that $m_i$ covers $S^1$ once under the projection of the fibering, and each $l_i$ lies in a fiber. It follows then that $m_i, l_i$ generate $\pi_1$ of $T_i$, but further, it also follows that each $m_i, l_i$ is not homotopic to 0 in $M_1$. Join each $T_i$ to a base point, $b$, in $M_1$ by an arc $\alpha_i$. Then by the above remarks each natural map $\pi_1(T_i \cup \alpha_i, b) \rightarrow \pi_1(M_1, b)$ is a monomorphism. Consider now the group $\pi_1(M_1, b) = G$ and subgroups $\pi_1(T_i \cup \alpha_i, b) = A_i$. If a different set of arcs $\alpha_i$ be selected, then a set of subgroups $\overline{A}_i$ results, where each $\overline{A}_i$ is a conjugate of $A_i$. In view of this, we may investigate the topological invariant $(G, [A_1], [A_2], \ldots, [A_n])$, where $G$ is $\pi_1(M_1, b)$ and $[A_i]$ is the conjugacy class containing $A_i$. Call this invariant the peripheral system of $M_1$. (See [5] for the source of this invariant.)

**Theorem 2.** Suppose a compact 3-manifold $M_2$ has peripheral system $(G', [A'_1], [A'_2], \ldots, [A'_n])$, then if there exists an isomorphism $\phi: G \rightarrow G'$, mapping $[A_i]$ onto $[A'_i]$, $M_1$ is homeomorphic to $M_2$.

**Proof.** By Stallings theorem [1], $M_2$ is fibered over $S^1$ with fiber a 2-manifold $S_2$, where $\pi_1(S_2) = \phi(\pi_1(S_1))$. Now define homeomorphisms $\Psi_i: T_i \cup \alpha_i \rightarrow U_i \cup \beta_i$ (where $U_i$ are the boundary tori of $M_1$ and $\beta_i$ are arcs joining $U_i$ to a base point in $M_2$) such that $\Psi^* = \phi$ for each element $x$ in $\pi_1(T_i \cup \alpha_i)$. This may be done by virtue of [2] and the hypothesis. It is no loss of generality to assume the $\alpha_i$ all lie on one fiber, and similarly the $\beta_i$. Nielsen’s [2] may be slightly generalized (as in [6]) so that a homeomorphism $\Psi_{n+1}$ may be constructed from the fiber containing the $\alpha_i$ to the fiber containing the $\beta_i$, satisfying $\Psi^* = \phi$ for elements $x$ in $H_1$, and agreeing with the $\Psi_i$ on $(T_i \cup \alpha_i) \cap$ (fiber containing $\alpha_i$). Call the homeomorphism now defined on $\partial M_1 \cup$ (a fiber), $\Psi$. $\Psi$ may be extended to a small closed product neighborhood of $\partial M_1 \cup$ (a fiber). Denote by $N$ this neighborhood, and, by $\Psi$ the homeomorphism thereon defined. Now $M_1 - (\text{int } N)$ is a solid torus of some genus (being fibered over an interval), and it is easily seen that $\Psi^*$ maps the kernel of the inclusion $\pi_1(\partial(M_1 - \text{int } N)) \rightarrow \pi_1(M_1 - \text{int } N)$ onto the kernel of $\pi_1(\partial(M_2 - \text{int } \Psi(N))) \rightarrow \pi_1(M_2 - \text{int } \Psi(N))$. (The argument is exactly that in [6], with $H_1$ taking the place of the commutator subgroup.) Hence (as in [6]) $\Psi$ may be extended to all of $M_1$ and the theorem is proved.

**Bibliography**

THE PRODUCT OF A NORMAL SPACE AND A METRIC SPACE NEED NOT BE NORMAL

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Communicated by Deane Montgomery, January 16, 1963

An old—and still unsolved—problem in general topology is whether the cartesian product of a normal space and a closed interval must be normal. In fact, until now it was not known whether, more generally, the product of a normal space $X$ and a metric space $Y$ is always normal. The purpose of this note is to answer the latter question negatively, even if $Y$ is separable metric and $X$ is Lindelöf and hereditarily paracompact.

Perhaps the simplest counter-example is obtained as follows: Take $Y$ to be the irrationals, and let $X$ be the unit interval, retopologized to make the irrationals discrete. In other words, the open subsets of $X$ are of the form $U \cup S$, where $U$ is an ordinary open set in the interval, and $S$ is a subset of the irrationals. It is known, and easily verified, that any space $X$ obtained from a metric space in this fashion is normal (in fact, hereditarily paracompact). Now let $Q$ denote the rational points of $X$, and $U$ the irrational ones. Then in $X \times Y$ the two disjoint closed sets $A = Q \times Y$ and $B = \{ (x, x) \mid x \in U \}$ cannot be separated by open sets. To see this, suppose that $V$ is a neighborhood of $B$ in $X \times Y$. For each $n$, let

$$U_n = \{ x \in U \mid (\{ x \} \times S_{1/n}(x)) \subset V \}.$$

1 Supported by an N.S.F. contract.

2 The usefulness of this space $X$ for constructing counterexamples was first called to my attention, in a different context, by H. H. Corson.