THE PRODUCT OF A NORMAL SPACE AND A METRIC SPACE NEED NOT BE NORMAL

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An old—and still unsolved—problem in general topology is whether the cartesian product of a normal space and a closed interval must be normal. In fact, until now it was not known whether, more generally, the product of a normal space $X$ and a metric space $Y$ is always normal. The purpose of this note is to answer the latter question negatively, even if $Y$ is separable metric and $X$ is Lindelöf and hereditarily paracompact.

Perhaps the simplest counter-example is obtained as follows: Take $Y$ to be the irrationals, and let $X$ be the unit interval, retopologized to make the irrationals discrete. In other words, the open subsets of $X$ are of the form $U \cup S$, where $U$ is an ordinary open set in the interval, and $S$ is a subset of the irrationals. It is known, and easily verified, that any space $X$ obtained from a metric space in this fashion is normal (in fact, hereditarily paracompact). Now let $Q$ denote the rational points of $X$, and $U$ the irrational ones. Then in $X \times Y$ the two disjoint closed sets $A = Q \times Y$ and $B = \{(x, x) \mid x \in U\}$ cannot be separated by open sets. To see this, suppose that $V$ is a neighborhood of $B$ in $X \times Y$. For each $n$, let

$$U_n = \{x \in U \mid (\{x\} \times S_{1/n}(x)) \subset V\},$$

1 Supported by an N.S.F. contract.

2 The usefulness of this space $X$ for constructing counterexamples was first called to my attention, in a different context, by H. H. Corson.
where \( S_{1/n}(x) \) denotes the \( 1/n \)-sphere about \( x \) in \( Y \). The \( U_n \) cover \( U \), and since \( U \) is not an \( F_\sigma \) in \( X \), there exists an index \( k \) such that \( U_k \cap Q \neq \emptyset \). Pick an \( x \) in \( U_k \cap Q \), and then pick \( y \in Y \) such that \( |x - y| < 1/2k \). Then \( (x, y) \in A \), and we need only show that any rectangular neighborhood \( R \times S \) of \( (x, y) \) in \( X \times Y \) intersects \( V \). To do that, pick \( x' \in R \cap U_k \) so that \( |x' - x| < 1/2k \). Then \( (x', y) \in R \times S \); also

\[
|x' - y| \leq |x' - x| + |x - y| < \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k},
\]

so \( (x', y) \in V \) because \( x' \in U_k \). That completes the proof.\(^3\)

The space \( X \) in the above example is neither Lindelöf nor separable. If Lindelöf is desired, let \( Y' \) be an uncountable subset of the unit interval, all of whose compact subsets are countable; such spaces exist \([1, p. 422]\). Letting \( X' \) be the unit interval, retopologized to make \( Y' \) discrete, we see just as before that \( X' \) is hereditarily paracompact and that \( X' \times Y' \) is not normal; moreover, because of the peculiar property of \( Y' \), it is easily checked that \( X' \) is Lindelöf.\(^4\) This \( X' \) is still not separable; it can, however, be embedded as a closed subset of a separable, Lindelöf, paracompact space \( X'' \), and then \( X'' \times Y' \) is also not normal.\(^5\)

Note that none of the above spaces \( X, X', \) and \( X'' \) are—or could be—perfectly normal, since the product of a paracompact, perfectly normal space and a metrizable space is known to be paracompact \([2, Proposition 5]\). That explains why none of our \( X' \)'s are either hereditarily Lindelöf, or separable and hereditarily paracompact, since—as is not hard to see—that would make them perfectly normal.

REFERENCES


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\( ^3 \) The above construction remains valid if \( Y \) is any separable metric space which is not \( \sigma \)-compact, or, even more generally, any metric space which can be embedded as a non-\( F_\sigma \) subset in another metric space. For instance, \( Y \) may be any infinite-dimensional Banach space.

\( ^4 \) If the continuum hypothesis is assumed, one can even find a Lindelöf, hereditarily paracompact space whose product with the irrationals is not normal.

\( ^5 \) Observe that both \( X \) and \( X' \)—but not \( X'' \)—have a \( \sigma \)-disjoint (and hence point-countable) base.