This follows from the continuity of \( F \), as before. The proof of the theorem is complete.

REFERENCES


INSTITUTE FOR DEFENSE ANALYSES AND
UNIVERSITY OF CHICAGO

TRANSVERSALITY IN MANIFOLDS OF MAPPINGS

BY RALPH ABRAHAM

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1. **Introduction.** Let \( X \) and \( Y \) be differentiable manifolds and \( \mathfrak{a} \) a space of mappings from \( X \) to \( Y \). A common problem in differential topology is to approximate a mapping in \( \mathfrak{a} \) by another in \( \mathfrak{a} \) which is transversal to a given submanifold \( W \subset Y \). Thus if \( \mathfrak{a}_{X,W} \) is the subspace of mappings transversal to \( W \) it is important to know if \( \mathfrak{a}_{X,W} \) is dense in \( \mathfrak{a} \). Some famous examples are the Whitney immersion and embedding theorems [8] and the Thom transversality theorem [4; 7]. In the next section we give sufficient conditions for density in case \( \mathfrak{a} \) is a Banach manifold. The proof of the density theorem is indicated in the third section, and in the final section the Thom transversality theorem is obtained as a corollary.

2. **Density theorems.** Throughout this section \( X \) will be a manifold with boundary, \( Y \) and \( Z \) manifolds, \( W \subset Y \) a submanifold (\( W, Y, Z \) without boundary) all of class \( C^r, r \geq 1 \), and modelled on Banach spaces (see [3] for definitions).

2.1. **DEFINITION.** A \( C^r \) mapping \( f: X \rightarrow Y \) is transversal to \( W \) at a point \( x \in X \) iff either \( f(x) \not\in W \), or \( f(x) = w \in W \) and there exists a neighborhood \( U \) of \( x \in X \) and a local chart \((V, \psi)\) at \( w \in Y \) such that

\[
\psi: V \rightarrow E \times F: V \cap W \rightarrow E \times 0,
\]

\( \pi_1 \circ \psi \) is a diffeomorphism of \( V \cap W \) onto an open set of \( E \), and \( \pi_2 \circ \psi \circ f \mid U \) is a submersion [3, p. 20], where \( \pi_1: E \times F \rightarrow E \) and \( \pi_2: E \times F \rightarrow F \).

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\[\pi_2: E \times F \to F\] are the projections. The mapping \(f\) is transversal to \(W\) on a subset \(K \subset X, f|K \pitchfork W,\) iff \(f\) is transversal to \(W\) at every point \(x \in K;\) and \(f\) is transversal to \(W, f \pitchfork W,\) iff \(f|X \pitchfork W.\)

For some basic properties of transversality see Lang [3, p. 22]. Suppose \(f: X \to Y\) and \(g: Z \to Y\) are \(C^r\) mappings, and let \(\Delta \subset Y \times Y\) denote the diagonal.

2.2. DEFINITION. The mappings \(f: X \to Y\) and \(g: Z \to Y\) are transversal at points \(x \in X\) and \(z \in Z\) iff the product \(f \times g: X \times Z \to Y \times Y\) is transversal to \(\Delta\) at \((x, z)\) in the sense of Definition 2.1. The mappings are transversal on sets \(K \subset X\) and \(M \subset Z, f|K \pitchfork g|M,\) iff
\[f \times g|K \times M \pitchfork \Delta,\]
and they are transversal, \(f \pitchfork g,\) iff \(f \times g \pitchfork \Delta.\)

2.3. DEFINITION. A \(C^r\) manifold of mappings from \(X\) to \(Y\) is a set \(\mathfrak{C}\) of \(C^r\) mappings from \(X\) to \(Y\) which is a \(C^r\) manifold such that the evaluation mapping
\[\text{ev}: \mathfrak{C} \times X \to Y: (f, x) \to f(x)\]
is of class \(C^r.\)

For example it is known that if \(X\) is compact then the set \(C^r(X, Y)\) of all \(C^r\) mappings from \(X\) to \(Y\) has a natural structure of \(C^r\) manifold of mappings [1, p. 31; 2; 5].

If \(K \subset X\) is any subset, and \(\mathfrak{C}\) a manifold of mappings from \(X\) to \(Y,\) let \(\mathfrak{C}_{K, w} \subset \mathfrak{C}\) be the subspace of mappings which are transversal to \(W\) on the set \(K.\) The following is easy to prove.

2.4. OPENNESS THEOREM. If \(K \subset X\) is a compact set, \(W \subset Y\) a closed submanifold, and \(\mathfrak{C} \subset C^r(X, Y)\) a \(C^r\) manifold of mappings, then the subset \(\mathfrak{C}_{K, w}\) is open in \(\mathfrak{C}.\)

Recall that a residual set in a topological space is a countable intersection of open dense sets, a Baire space is one in which every residual set is dense, and by the Baire category theorem every Banach manifold is a Baire space.

2.5. DENSITY THEOREM. Let \(X\) be an \(n\)-manifold with boundary, \(K \subset X\) any subset, \(Y\) a Banach manifold (without boundary), and \(W \subset Y\) a closed submanifold (without boundary) of finite codimension \(q,\) all of class \(C^r.\) Let \(\mathfrak{C} \subset C^r(X, Y)\) be a \(C^r\) manifold of mappings. If the evaluation map of \(\mathfrak{C}\)
\[\text{ev}: \mathfrak{C} \times X \to Y: (f, x) \to f(x)\]
is transversal to \(W\) on \(K\) and \(r \geq \max(n - q, 0),\) then \(\mathfrak{C}_{K, w} \subset \mathfrak{C}\) is residual.
This theorem is the main result, and may be generalized in several ways. For example suppose $X$, $Y$ and $Z$ are finite dimensional, $\mathcal{A} \subset C^r(X, Y)$ and $\mathcal{B} \subset C^r(Z, Y)$ are $C^r$ manifolds of mappings, and $ev_{\mathcal{A}}$ and $ev_{\mathcal{B}}$ are the respective evaluation mappings. If $K \subset X$ and $M \subset Z$, let $\mathcal{A} \times \mathcal{B}_{\mathcal{K} \times \mathcal{M}} = \{(f, g) \in \mathcal{A} \times \mathcal{B} : f|K \cap g| M\}$. Then the following are immediate.

2.6. **COROLLARY.** If $ev_{\mathcal{A}}|\mathcal{A} \times K \cap ev_{\mathcal{B}}|\mathcal{B} \times M$ and $r > \max(\dim X + \dim Z - \dim Y, 0)$, then $\mathcal{A} \times \mathcal{B}_{\mathcal{K} \times \mathcal{M}} \subset \mathcal{A} \times \mathcal{B}$ is residual.

Let $\mathcal{B} = \{g\}$, a single map, and

$$\mathcal{A}_{\mathcal{K}, g} = \{f \in \mathcal{A} : f|K \cap g\}.$$

2.7. **COROLLARY.** If $ev_{\mathcal{A}}|\mathcal{A} \times K \cap g$ and

$$r > \max(\dim X + \dim Z - \dim Y, 0),$$

then $\mathcal{A}_{\mathcal{K}, g} \subset \mathcal{A}$ is residual.

If in 2.7 $g$ is an embedding and $W$ its image, then 2.5 is obtained with the condition "$W$ closed" deleted.

Note the symmetry of 2.2 and 2.6. Both may be extended to $n$-tuples of mappings having a common target, and the symmetry can be completed by allowing all sources to be manifolds with boundary, making only trivial modifications.

3. **Proof of the density theorem.** The Density Theorem 2.5 is proved from the following lemma by an easy point set argument.

**DENSITY LEMMA.** If $X$ has finite dimension $n$, $W \subset Y$ is a closed submanifold having finite codimension $q$, $\mathcal{A} \subset C^r(X, Y)$ is a $C^r$ manifold of mappings with $r > \max(n - q, 0)$, and the evaluation mapping is transversal to $W$ at a point $(f, x) \in \mathcal{A} \times X$, then there exists a neighborhood $U$ of $f \in \mathcal{A}$ and a neighborhood $V$ of $x \in X$ such that $W_{f, W} \subset U$ is dense.

If $f(x) \in W$ the lemma is trivial. If $f(x) = w \in W$ the proof is immediate from these three propositions.

**PROPOSITION A.** If the evaluation map is transversal to $W$ at $(f, x)$ and $f(x) = w \in W$, there are neighborhoods $U$ of $f \in \mathcal{A}$ and $V$ of $x \in X$ such that every point $g \in U$ is contained in a $p$-dimensional submanifold $\sum^p$, $0 \leq p \leq q$, such that $ev|\sum^p \times V \cap W$.

The proof of this proposition is prosaic, relying on techniques which have become standard since the publication of Lang's book [3].

For the second proposition suppose $\sum^p$ is a $p$-dimensional sub-
manifold of \( \alpha, V \) an open set of \( X \), and \( \xi = \text{ev}\left| \sum^p V \right. \) is transversal to \( W \). Then \( W' = \xi^{-1}(W) \) is a submanifold of codimension \( q \) of \( \sum^p V \). Let \( \sigma: W' \to \sum^p \) denote the restriction to \( W' \) of the projection \( \sum^p V \to \sum^p \).

**Proposition B.** If \( \sigma \) is transversal to a point \( f \in \sum^p \), then \( f \) is transversal to \( W \) on \( V \).

The proof of this proposition is a straightforward interpretation of the definitions.

The third proposition is a well-known theorem of Sard \([6]\). Recall that if \( f: X \to Y \) is any \( C^1 \) mapping, a point \( y \in Y \) is a critical value of \( f \) iff it is false that \( f \cap \{ y \} \). Let \( \Omega_f \) be the set of all critical values of \( f \).

**Proposition C (Sard).** If \( f: \mathbb{R}^s \to \mathbb{R}^t \) is of class \( C^r \) with \( r > \max(s-t, 0) \), then \( \Omega_f \subset \mathbb{R}^t \) has outer measure zero.

4. **The Thom transversality theorem.** Let \( X \) be a \( C^r \) manifold with boundary, \( Y \) a \( C^r \) manifold (without boundary), \( \pi^k: J^k(X, Y) \to X \times Y \) the \( k \)-jet bundle of \( C^k \) maps from \( X \) to \( Y \), \( 0 \leq k \leq r \), and

\[
j^k: C^r(X, Y) \to C^{r-k}(X, J^k(X, Y))
\]

the \( k \)-jet extension, where \( C^r(X, Y) \) has the \( C^r \) topology of compact convergence (a Baire space). Let \( W \) be a \( C^{r-k} \) manifold (without boundary), \( F \in C^{r-k}(W, J^k(X, Y)) \), and

\[
C^r_p(X, Y) = \{ f \in C^r(X, Y): j^k f \cap F \}.
\]

Finally, suppose \( W, X \) and \( Y \) are finite dimensional.

**Jet transversality theorem (Thom).** If

\[
r > \max\{ \dim X + \dim W - \dim J^k(X, Y), 0 \},
\]

then the subspace \( C^r_p(X, Y) \subset C^r(X, Y) \) is residual for every \( C^{r-k} \) mapping \( F: W \to J^k(X, Y) \).

**Proof.** First suppose \( X \) is compact. Then it is known that \( \alpha = j^k[C^r(X, Y)] \) has a natural structure of \( C^{r-k} \) manifold of mappings compatible with the topology of compact convergence \([1]\). Furthermore a standard computation using a local chart and a \( C^r \) characteristic function shows that the differential of the evaluation mapping

\[
ev: \alpha \times X \to J^k(X, Y): (j^k f, x) \to j^k f(x)
\]

is surjective at any point \( (j^k f, x) \in \alpha \times X \), so the evaluation map is transversal to any mapping \( F: W \to J^k(X, Y) \). Thus if
$r > \max \{ \dim X + \dim W - \dim J^k(X, Y), 0 \}$, the Openness and Density Theorems 2.4 and 2.7 imply that $\alpha_{X,F} \subset \alpha$ is open and dense. Now suppose $X$ is not compact. Then, using a countable covering of $X$ by compact manifolds with boundary, a simple point set argument, and the proof above, $\alpha_{X,F} \subset \alpha$ is seen to be residual. But $j^*: C^r(X, Y) \to \alpha$ is a homeomorphism so $C^r_F(X, Y) = j^*^{-1}(\alpha_{X,F})$ is residual in $C^r(X, Y)$.

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COLUMBIA UNIVERSITY