TWO THEOREMS CONCERNING FUNCTIONS
HOLOMORPHIC ON MULTIPLE
CONNECTED DOMAINS

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1. Let $\Omega$ be a finitely connected plane domain whose boundary, 
$\partial \Omega$, consists of the circles $\Gamma_0, \Gamma_1, \cdots, \Gamma_n$. We assume $\Gamma_j$ lies in the 
interior of $\Gamma_0$ for $j = 1, 2, \cdots, n$. Let $\Delta_0$ be the interior of $\Gamma_0$ and let 
$\Delta_j$ be the exterior of $\Gamma_j$, $j = 1, 2, \cdots, n$. We then have $\Omega = \bigcap_{j=0}^{n} \Delta_j$. 
Let $H_\infty[\Omega]$ be the collection of all bounded holomorphic functions in 
$\Omega$. We shall say that a set $S$ of points of $\Omega$ is an interpolation set for $\Omega$ 
if given a bounded complex valued function $w$ on $S$ there is $f \in H_\infty[\Omega]$ 
such that $f(z) = w(z)$ for all $z \in S$. If $\{z_n\}_{n=1}^\infty$ is a sequence in $\Omega$, without 
limit points in $\Omega$, we write $\{z_n\} = S_0 \cup S_1 \cup \cdots \cup S_n$ where the $S_j$ are 
pairwise disjoint and where the only limit points of $S_j$ lie in $\Gamma_j$, 
j = 0, 1, \cdots, n.

In the present note we sketch proofs for the following two theo­
rems:

THEOREM A. The sequence $\{z_n\}$ is an interpolation set for $\Omega$ if and 
only if each $S_j$ is an interpolation set for the disc $\Delta_j$.

THEOREM B. Let $f_1, f_2, \cdots, f_m$ be functions in $H_\infty[\Omega]$ such that 
$|f_1(z)| + |f_2(z)| + \cdots + |f_m(z)| \geq \delta > 0$ for all $z \in \Omega$. Then there exist 
functions $g_1, g_2, \cdots, g_m \in H_\infty[\Omega]$ such that $f_1g_1 + f_2g_2 + \cdots + f_mg_m = 1$.

L. Carleson [2] has established Theorem B in case $\Omega$ is the open 
unit disc. He has also proved [1] that the sequence $\{z_n\}_{n=1}^\infty$ is an 
interpolation sequence for the open unit disc if and only if there is a 
$\delta > 0$ such that

$$\prod_{n \neq k} \left| \frac{z_n - z_k}{1 - \overline{z}_nz_k} \right| > \delta$$

for $k = 1, 2, 3, \cdots$. For a discussion and alternative proof see [3, pp.
194–208].

2. Outline of the proof of Theorem A. Let $B_j$ be the Blaschke 
product associated with the disc $\Delta_j$ and the set of points $S_j$, 
$j = 0, \cdots, n$. Note that there is an $\eta > 0$ such that $|B_j(z)| > \eta$ for 
z $\in S_k$ if $k \neq j$.

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Suppose now that \( S_j \) is an interpolation set for \( \Delta_j, j = 0, \cdots, n \). Let \( w \) be a bounded function on \( S \), and let \( f_j \in \mathcal{H}_\infty[\Delta_j] \) be such that
\[
f_j(z) = w(z)/(B_0(z) \cdots B_{j-1}(z)B_{j+1}(z) \cdots B_n(z))
\]
for all \( z \in S_j \). Define
\[
F = f_0B_1B_2 \cdots B_n + f_1B_0B_2 \cdots B_n + \cdots + f_nB_0B_1 \cdots B_{n-1}.
\]
Then \( F \in \mathcal{H}_\infty[\Omega] \) and \( F(z) = w(z) \) for all \( z \in S \).

Conversely, assume that \( \{z_n\}_{n=1}^\infty \) is an interpolation set for \( \Omega \). If \( f \in \mathcal{H}_\infty[\Omega] \) we define \( ||f|| \) by
\[
(1) \quad ||f|| = \sup \{|f(z)| : z \in \Omega \}.
\]
A Banach space argument like that in [3, p. 196] shows that there is a constant \( M \) such that if \( w \) is a function on \( \{z_n\}_{n=1}^\infty \) with \( |w(z)| \leq 1 \) for all \( z \in \{z_n\}_{n=1}^\infty \), then there is \( f \in \mathcal{H}_\infty[\Omega] \) with \( ||f|| \leq M \) and \( f(z) = w(z), \ z \in \{z_n\}_{n=1}^\infty \). Given \( z_k \in S_j \), let \( B_j^{(k)} \) be the Blaschke product associated with the disc \( \Delta_j \) and the set \( S \setminus \{z_k\} \). Let \( f \in \mathcal{H}_\infty[\Omega] \) be such that \( f(z_n) = 0 \) for \( n \neq k, f(z_k) = 1 \) and such that \( ||f|| \leq M \). The function
\[
g = f/(B_0 \cdots B_{j-1}B_j^{(k)}B_{j+1} \cdots B_n)
\]
is in \( \mathcal{H}_\infty[\Omega] \). Since there is \( \delta > 0 \) such that \( |B_i(z)| \geq \delta \) for all \( z \in \Gamma_j, i \neq j \), we have that \( ||g|| \leq M/\delta^n \). In particular then \( |g(z_k)| \leq M/\delta^n \). This yields
\[
|\delta^nM^{-1}f(z_k)/(B_0(z_k) \cdots B_{j-1}(z_k)B_{j+1}(z_k) \cdots B_n(z_k))| \leq |B_j^{(k)}(z_k)| .
\]
Since \( f(z_k) = 1 \), and the product \( B_0 \cdots B_{j-1}B_{j+1} \cdots B_n \) is uniformly bounded away from zero on \( S_j \), we have that \( B_j^{(k)}(z_k) \geq \delta_1 > 0 \). This estimate is uniform in \( k \), so \( S_j \) is an interpolation set for \( \Delta_j \).

3. Outline of the proof of Theorem B. Observe that \( \mathcal{H}_\infty[\Omega] \) is a commutative Banach algebra with identity if it is given the norm defined by (1). Let \( \mathfrak{M}[\Omega] \) be the maximal ideal space of \( \mathcal{H}_\infty[\Omega] \); we regard \( \mathfrak{M}[\Omega] \) as the collection of all nonzero complex homomorphisms of \( \mathcal{H}_\infty[\Omega] \) with the weak* topology. Let \( \mathfrak{M}_e[\Omega] \) be the collection of those homomorphisms \( \phi_\lambda \) of the form \( \phi_\lambda(f) = f(\lambda), \ \lambda \in \Omega \). It is known [3, p. 163] that to establish our result it suffices to prove \( \mathfrak{M}_e[\Omega] \) dense in \( \mathfrak{M}[\Omega] \).

For \( j = 1, 2, \cdots, n \), let \( \mathcal{H}_\infty^0[\Delta_j] \) be the closed subalgebra of \( \mathcal{H}_\infty[\Omega] \) consisting of those \( f \) which are restrictions to \( \Omega \) of functions in \( \mathcal{H}_\infty[\Delta_j] \) which vanish at infinity. It is known [4, p. 56] that if \( f \in \mathcal{H}_\infty[\Omega] \), then \( f \) can be written in the form
(2) \[ f = f_0 + f_1 + \cdots + f_n, \]
\[ f_0 \in H_\infty[\Delta_0], f_j \in H_\infty^0[\Delta_j], \ 1 \leq j. \]
It is immediate that this decomposition is unique; it yields
\[ (3) \ H_\infty[\Omega] = H_\infty[\Delta_0] \oplus H_\infty^0[\Delta_1] \oplus \cdots \oplus H_\infty^0[\Delta_n], \]
the direct sum being understood in the sense of Banach spaces.

Following some ideas of I. J. Schark (see [3, p. 159, ff.]), we note that the function \( z \) is in \( H_\infty[\Omega] \). It gives rise to the function \( \xi \) on \( T[\Omega] \) given by \( \xi(\phi) = \phi(z) \). We can prove that \( \xi \) maps \( T[\Omega] \) onto \( \Omega \) and that \( \xi \) is one-to-one over \( \Omega \). If \( \alpha \in \partial \Omega \), set \( M_\alpha = \{ \phi \in T[\Omega] : \xi(\phi) = \alpha \} \). A slight modification of the argument for the disc case shows that if \( f \in H_\infty[\Omega] \), then \( f \) is constant on \( T[\Omega] \{ \alpha \} \) and that if \( f \) is so extensible, then \( f(z) = f(\alpha) \) for all \( \phi \in T[\Omega] \).

Suppose now that \( \phi \) is a multiplicative linear functional defined on \( H_\infty[\Delta_0] \) viewed as a subalgebra of \( H_\infty[\Omega] \) by the direct sum decomposition (3). Let \( \phi(z) \in \Omega \). Then \( \phi \) admits a unique extension to an element of \( T[\Omega] \). This is clear since \( \xi \) maps \( T[\Omega] \) onto \( \Omega \) and is one-to-one over \( \Omega \). If \( \alpha = \phi(z) \) lies in \( \Gamma_0 \), \( \phi \) also admits a unique extension to an element of \( T[\Omega] \). For uniqueness, suppose that \( \phi^* \) is an extension of \( \phi \) to all of \( H_\infty[\Omega] \). For \( f \in H_\infty[\Omega] \), write \( f = f_0 + f_1 + \cdots + f_n \) in accordance with (2). The linearity of \( \phi^* \) implies that \( \phi^*(f) = \phi^*(f_0) + \phi^*(f_1) + \cdots + \phi^*(f_n) \). Since \( \phi^* \) is an extension of \( \phi \), and since, for \( j = 1, 2, \ldots, n \), \( f_j \) is continuously extensible to \( \Omega \cup \{ \alpha \} \), it follows that \( \phi^*(f) = \phi(f_0) + f_1(\alpha) + \cdots + f_n(\alpha) \). This establishes the uniqueness of the extension. This choice of \( \phi^* \) yields a multiplicative functional. To see this, suppose \( g \in H_\infty[\Omega] \) and write \( g = g_0 + g_1 + \cdots + g_n \) by (2). Then \( fg = \sum_{j,k} f_j g_k \). Since \( \phi^* \) is plainly linear, we need only show \( \phi^*(f_j g_k) = \phi^*(f_j) \phi^*(g_k) \). If neither \( j \) nor \( k \) is zero, \( f_j g_k \) is continuously extensible to \( \Omega \cup \{ \alpha \} \), so we need only consider terms of the form \( f_0 g_k \) and \( f_j g_0 \). Suppose then that \( f \in H_\infty[\Delta_0], g \in H_\infty[\Delta_j], j \neq 0 \). Since \( \phi^*(g) = g(\alpha) \), we are finished if we can show \( \phi^*(fg - g(\alpha)f) = 0 \).

Write
\[ fg - g(\alpha)f = h_0 + h_1 + \cdots + h_n \]
in accordance with (2). Then \( h_j \) is continuous at \( \alpha \) for \( j = 1, \ldots, n \), and since \( fg - g(\alpha)f \) is continuous at \( \alpha \), it follows that \( h_0 \) must be continuous at \( \alpha \) so that \( \phi(h_0) = h_0(\alpha) \). Therefore \( \phi^*(fg - g(\alpha)f) = h_0(\alpha) + h_1(\alpha) + \cdots + h_n(\alpha) = 0 \). We conclude that \( \phi^* \) is multiplicative.

If \( \phi \) is a multiplicative linear functional on \( H_\infty[\Delta_0] \) such that \( \phi(z) \in \Gamma_j \) for \( j \neq 0 \), our argument indicates that \( \phi \) admits many exten-
sions to an element of $\mathcal{M}[\Omega]$. If $\phi(z) \in \Delta_0 \setminus \overline{\Omega}$, then $\phi$ admits no extension.

The same argument applies to $\Delta_1, \ldots, \Delta_n$ in place of $\Delta_0$. This also shows that every element of $\mathcal{M}[\Omega]$ is determined by its action on the subalgebras $H^\infty_\alpha[\Delta_0], H^0_\alpha[\Delta_1], \ldots, H^0_\alpha[\Delta_n]$. It now follows that $\mathcal{M}_e[\Omega]$ is dense in $\mathcal{M}[\Omega]$. For suppose $\phi \in \mathcal{M}[\Omega]$, and suppose $\alpha = \phi(z) \in \Gamma_k$. Let $\phi^{(j)}$ be the restriction of $\phi$ to the subalgebra $H^\infty_\alpha[\Delta_j]$. By Carleson's result for the disc, there is a point $\lambda \in \Delta_k$ such that the point evaluation $\phi^{(k)}(\lambda)$ near $\phi^{(k)}$ in the sense of the weak* topology in the maximal ideal space of $H^\infty_\alpha[\Delta_k]$. If $\lambda$ is near $\alpha$, then $\lambda \in \Omega$, and each of the point evaluations at $\lambda, \phi^{(j)}(\lambda)$ for $j \neq k$ is near the point evaluation $\phi^{(k)}(\lambda)$ in the maximal ideal space of $H^\infty_\alpha[\Delta_k]$. But then the point evaluation $\phi^{(k)}(\lambda) \in \mathcal{M}_e[\Omega]$ is near the homomorphism $\phi$ in $\mathcal{M}[\Omega]$. Thus $\mathcal{M}_e[\Omega]$ is dense in $\mathcal{M}[\Omega]$, and we have our result.

4. We can relax our condition on the boundary of $\Omega$ as follows. Our results are plainly invariant under conformal mapping. It is known [5, p. 377] that every finitely connected domain with no nondegenerate boundary components is conformally equivalent to a domain bounded by circles. Thus our results apply to this more general class of domains.

References


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