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SIMPLY INVARIANT SUBSPACES

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Let $L^1$, $L^2$ denote respectively the spaces of summable and square summable functions on the circle group and $H^1$, $H^2$ their subspaces consisting of those functions whose Fourier coefficients vanish for negative indices. A closed subspace $M$ of $L^1$ or $L^2$ is "invariant" if

$$ \chi M \subseteq M $$

and "simply invariant" if the above inclusion is strict, where $\chi$ is the character

$$ \chi(x) = e^{ix}. $$

The structure of simply invariant subspaces is known, namely, they are precisely the subspaces of the form $qH^1$ or $qH^2$ (respectively) where $q$ is a measurable function of modulus 1 a.e. Beurling [1] first proved this for subspaces $M \subseteq H^2$; for $M \subseteq H^1$, this is due to de Leeuw-Rudin [5]; for $M \subseteq L^1$, due to Helson-Lowdenslager [3] and for $M \subseteq L^1$, due to Forelli [2]. In [3] Helson-Lowdenslager also gave a simple proof of the $H^2$ case, free of function theoretic considerations. Using their arguments Hoffman [4] extended this result to simply invariant subspaces of $H^2(dm)$ defined over logmodular algebras. In this paper we prove this result for simply invariant subspaces of $L^2(dm)$ and $L^1(dm)$ over logmodular algebras; the results of the previous authors follow as a corollary. The proofs of the previous authors

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1 This work was done while I held a visiting appointment at the University of California, Berkeley.

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do not extend to this general case as they depend on facts which either have no analogues or are not true for the logmodular algebras; when specialised to their contexts, our proof turns out to be even simpler. Our proof for the case of $L^2(dm)$ was inspired by that of Helson-Lowdenslager for the $H^2$ case and is in the same spirit as theirs.

Let $X$ be a compact Hausdorff space and $A$ a subalgebra of the algebra $C(X)$ of complex continuous functions on $X$ with the uniform norm.

$A$ is logmodular if

i. $A$ is uniformly closed,

ii. $A$ contains the constant functions,

iii. $A$ separates the points of $X$ and

iv. the set of functions $\log |f|$ where $f, 1/f \in A$, is uniformly dense in the algebra of real continuous functions on $X$.

Let $m$ be a probability Baire measure on $X$ which is "multiplicative" on $A$, meaning

$$\int fg \, dm = \int f \, dm \int g \, dm$$

for all $f, g \in A$ (such measures always exist), and let $H^1(dm), H^2(dm)$ denote the closures of $A$ in $L^1(dm), L^2(dm)$ respectively. The invariant subspaces $M$ are now closed subspaces of $L^1(dm), L^2(dm)$, which are invariant under multiplication by functions in $A$ or equivalently by functions in $A_0$, where

$$A_0 = \{ f \mid f \in A, \int f \, dm = 0 \}$$

and the simply invariant $M$'s are those for which the inclusion $A_0M \subset M$ is strict.$^2$

In the case considered earlier, $X$ was the unit circle, $A_0$ was the uniform closure of the algebra generated by $\chi$ in $C(X)$ and $m$ the normalised Lebesgue measure. We have

**Theorem.**

1. The simply invariant subspaces of $L^2(dm)$ are precisely the subspaces of the form $qH^2(dm)$ where $q \in L^2(dm)$ and $\| q \| = 1$ a.e. $(dm)$.

2. The simply invariant subspaces of $L^1(dm)$ are precisely the subspaces of the form $qH^1(dm)$ where $q \in L^1(dm)$ and $\| q \| = 1$ a.e. $(dm)$.$^3$

$^2$ $A_0M$ should be replaced by its closure in $L^2(dm)$ respectively $L^1(dm)$, which necessitates changes in the proof.

$^3$ The details of the proof of the $L^1$ theorem and its function theoretic consequences will be published separately.
PROOF. It is obvious that subspaces of the form $qH^2(dm)$, $gH^1(dm)$ are invariant; they are simply invariant because for instance, $q \in qH^2(dm)$, $qH^1(dm)$ while $q \notin qA_0H^2(dm)$, $qA_0H^1(dm)$. To prove the converse:

1. We need the following facts about logmodular algebras [4, pp. 284, 293]:
   (a) $A + \overline{A}$ is dense in $L^2(dm)$ where the bar denotes complex conjugation,
   (b) if $\mu$ is any positive Baire measure on $X$ such that $\int f \, d\mu = 0$ for all $f \in A_0$ then $d\mu = c \, dm$ for some constant $c$.

Now let $M \subseteq L^2(dm)$ be simply invariant and let $q \in M \Theta A_0M$, $q \neq 0$. Then $q \perp A_0g$, so $\int |q|^2 \, dm = 0$ for all $f \in A_0$ and by (b), $|q|^2 = c$ a.e. By modifying $q$ we may assume that $|q| = 1$ a.e.

Clearly $qH^2(dm) \subseteq M$, because of invariance of $M$. Let $g \in M \Theta qH^2(dm)$. Then $g \perp qA$, so $g \perp A$. Also $A_0g \subseteq A_0M$, so $q \perp A_0g$ so that $g \perp A_0$. Thus $g \perp A + \overline{A}$, hence $g = 0$ a.e. by (a) and since $|q| = 1$ a.e., $g = 0$. Thus $M = qH^2(dm)$.

2. We use (1) to prove (2). Let $N \subseteq L^1(dm)$ be simply invariant and let $M = N \cap L^2(dm)$. $M$ is clearly an invariant subspace of $L^2(dm)$. We shall show that it is actually simply invariant. Let $f \in N$. We can find $f_1, f_2 \in L^2(dm)$ such that $f = f_1f_2$; we may also assume that one of them, say, $f_2$ is nonzero a.e. Then $f_2H^2(dm)$ is a simply invariant subspace of $L^2(dm)$ and is by (1) of the form $q_2H^2(dm)$, $|q_2| = 1$ a.e. Now

$$f_1q_2 \in f_1q_2H^2(dm) = f_1f_2H^2(dm) = fH^2(dm) \subseteq N.$$ 

Also $f_1q_2 \in L^2(dm)$. Hence $f_1q_2 \subseteq M$. Suppose $M = A_0M$. Then $f_1q_2 \in A_0M$. Let

$$f_1q_2 = f_0g, \quad f_0 \in A_0, \quad g \in M \subseteq N$$

and

$$f_2 = qgh, \quad h \in H^2(dm).$$

Then

$$f = f_1f_2 = f_1q_2h = f_0gh \in A_0NH^2(dm) \subseteq A_0N$$

and it follows that $N = A_0N$. Hence if $N$ is simply invariant, so is $M$.

Let then $M = qH^2(dm)$ by (1). We shall show that $N = qH^1(dm)$. Clearly $qH^2(dm) \subseteq N$. Let $f \in N$ and $f_1, f_2, q_3, h$ be as above. Then $f_1q_2 \in M = qH^2(dm)$. Let $f_1q_2 = gh'$, $h' \in H^2(dm)$. Then

$$f = f_1f_2 = f_1q_2h = qh'h \in qH^2(dm)$$

as $h', h \in H^2(dm)$. It follows that $N = qH^1(dm)$. 

We may remark that if $M \subseteq H^1(dm)$ is invariant and we assume with Hoffman [4, p. 293] that $fgdm \neq 0$ for at least one $g \in M$ then $M$ is certainly simply invariant and Hoffman's result follows. But this latter condition is not necessary for simple invariance as the example of $a^k H^1$, $k \geq 1$ shows.

REFERENCES


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