FLOWS ON SOLVMANIFOLDS

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Let $G$ be a connected, simply connected solvable Lie group and let $C$ be a closed subgroup such that $G/C$ is compact. Further, let $g(t)$ be a one parameter subgroup of $G$. Then $g(t)$ induces, acting to the left on $G/C$, a one parameter group of transformations. We will call $G/C$ a representation of a compact solvmanifold and the flow generated by $g(t)$ a $G$-induced flow. In [1] and [2] results concerning special $G$-induced flows were discussed in detail or research announced. It is the purpose of this note to state one aspect of the problem of $G$-induced flows and outline the solution that is now available. Full details and discussions will be presented elsewhere.

Let $G/C$ be a representation of a compact solvmanifold. Our main problem may be stated as follows: Give a necessary and sufficient condition for the existence of an ergodic $G$-induced flow on $G/C$. We will now outline our solution to this problem.

Algebraic preliminaries. Let $L(G)$ denote a solvable Lie algebra which may be taken over the real or complex fields. Then $L(G)$ is said to be of Type (R) if all the roots of the algebra are either 0 or pure imaginary. One of the main algebraic facts we will need is the following:

**Theorem.** Given a solvable Lie algebra $L(G)$ there exists a unique minimal ideal $H$ such that $L(G)/H$ is Type (R).

We will also need the companion concept of a really regular element of $L(G)$. $X \in L(G)$ will be called really regular if it is

1. regular,
2. $\text{ad}(X)$ has a maximal number of eigenvalues with nontrivial real part. The set of really regular elements is dense in $L(G)$.

**Theorem.** $G/C$ has a $G$-induced ergodic flow if and only if there is a really regular element $X$ of $L(G)$ such that $\exp(tX)$ induces an ergodic flow.

**Reduction Theorem.** A really regular element in $L(G)$ induces an ergodic flow on $G/C$ if and only if it induces an ergodic flow on $\text{Cl}(G/C)$, where $H$ is the unique minimal analytic normal subgroup of $G$ such that $G/H$ is Type (R) and $\text{Cl}$ denotes the closure operation.

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Hence we see that we have reduced the problem to the Type (R) case.

More algebra I. Let $S$ be a Type (R) solvable group, which is connected and simply connected. Then there exists a unique nilpotent analytic group $N$ and a compact abelian group $T$ of automorphisms of $N$ such that

(a) $S \subseteq T \cdot N$.

(b) Let $\varphi: S \to N$ be the projection mapping.

Then $\varphi$ is a homeomorphism of $S$ onto $N$.

(c) $S$ and $N$ generate $T \cdot N$.

We will call $T \cdot N$ the minimal splitting of $S$.

More algebra II. Let $C$ be a closed subgroup of a connected, simply connected Type (R) solvable group $S$. Let $T \cdot N$ be a minimal splitting for $S$. Then $S/N \cap C$ is compact and $C/N \cap C$ is a finite group. Further, the projection mapping $\varphi: S \to N$ induces a homeomorphism of $S/N \cap C$ onto $N/N \cap C$.

DEFINITION. We will say that a one-parameter group in $S$ is in general position if its projection onto $T$ is dense in $T$, where $T \cdot N$ is the minimal splitting of $S$. Then one can easily see that the image of a really regular element of $G$ is in general position in $S$ and conversely, given an element of $S$ in general position there exist really regular elements of $G$ which project onto it.

SECOND REDUCTION THEOREM. Let $S = G/H$, where $H$ is the unique minimal analytic normal subgroup of $G$ such that $G/H$ is Type (R). A $G$-induced flow on $S/C$ is ergodic if and only if there exists an $S$-induced flow which is ergodic on $S/N \cap C$.

THEOREM. Let $S$ be of Type (R). There exists an $S$-induced ergodic flow on $S/N \cap C$ if and only if the null space of $(t-I)$, for all $t \in T$, contains a vector in general position in $N/C_0[N, N]$ relative to the lattice $C/C_0([N, N] \cap C)$, where $C_0$ is the identity component of $C$.

REFERENCES


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