1. Introduction. It is well known that a transformation $T$ which preserves a finite measure has the mixing property

$T^{(k)} = T \times T \times \cdots \times T$ (k times, $k \geq 2$) is ergodic if and only if $T$ is weakly mixing \[1\].

The purpose of this note is to give, for each positive integer $k$, an example of a transformation $T$ which preserves a $\sigma$-finite infinite measure with the property,

$T^{(k)}$ is ergodic but $T^{(k+1)}$ is not ergodic.

We also give an example of a transformation $T$ which preserves a $\sigma$-finite infinite measure with the property

$T^{(k)}$ is ergodic for each $k = 1, 2, \ldots$.

A transformation $T$ with property (1.2) is said to have ergodic index $k$ and a transformation $T$ with property (1.3) is said to have infinite ergodic index. For completeness, we say that a nonergodic transformation has zero ergodic index.

Thus, for each $k = 0, 1, 2, \ldots, \infty$, infinite measure preserving transformations exist with ergodic index $k$, unlike finite measure preserving transformations which assume ergodic indices 0, 1, $\infty$ only.

The examples are taken from Gillis \[2\], and are Markov transformations derived from “centrally biased random-walks.”

2. Markov transformations preserving a $\sigma$-finite infinite measure. Let

$P = ||p(i, j)||$, \quad $i, j = 0, \pm 1, \pm 2, \cdots$

be a stochastic matrix with only one ergodic class, i.e.,

$p(i, j) \geq 0$, \quad $\sum_{j=-\infty}^{\infty} p(i, j) = 1$,

and for each $(i, j)$ there exists $n > 0$ for which $p^n(i, j) > 0$ where $P^n = ||p^n(i, j)||$. Assume also that there exists a left eigenvector

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INFINITE MEASURE PRESERVING TRANSFORMATIONS

\( \Lambda = \{ \lambda(i) \} \) (with eigenvalue one) with positive entries such that

\[ \sum_{i=0}^{\infty} \lambda(i) = \infty. \]

Let \( Z \) be the set of all integers and let

\[ X = \prod_{i=-\infty}^{\infty} Z_i, \quad Z_i = Z, \quad i = 0, \pm 1, \ldots. \]

A generic element of \( X \) is a point \( x = \{ z_i(x) \} \).

A cylinder of \( X \) is a set of the form

\[ C_{m,n}(x) = \{ y \in X : z_i(y) = z_i(x), m \leq i \leq n \}. \]

Let \( \mathcal{B} \) be the Borel field generated by the cylinders of \( X \) and let \( \mu \) be the \( \sigma \)-finite measure generated by the cylinder function

\[ \mu_{C_{m,n}}(x) = \lambda(z_m(x)) \prod_{i=m}^{n-1} \mu(z_i(x), z_{i+1}(x)). \]

It is clear that the measure \( \mu \) is invariant under the shift transformation \( T \),

\[ T \{ z_i \} = \{ z'_i \}, \quad z'_i = z_{i+1}, \]

and \( X = \bigcup_{i=-\infty}^{\infty} X_i, \quad \mu(X_i) = \lambda(i), \quad \mu(X) = \infty, \) where \( X_i = \{ x \in X : z_0(x) = i \} \).

We refer to \( (X, \mathcal{B}, \mu, T) \) as the \( \sigma \)-finite stationary Markov chain defined by \( \mu \). \( T \) is the Markov transformation defined by \( \mu \).

We shall be interested in the following conditions on \( \mu \):

I. For every \( i_1, \ldots, i_k \) there exists \( n > 0 \) such that

\[ \mu^n(i_1, j_1) \times \cdots \times \mu^n(i_k, j_k) > 0. \]

II. \[ \sum_{n=1}^{\infty} [\mu^n(0, 0)]^k = \infty. \]

**Theorem.** \( \mu \) satisfies I_\( k \) and II_\( k \) if and only if the Markov transformation \( T \) defined by \( \mu \) satisfies: \( T^{(k)} \) is ergodic with respect to \( \mu^{(k)} = \mu \times \cdots \times \mu \) (\( k \) times).

The above theorem can be deduced from a similar theorem in [3]. We indicate below the main points of the proof.

The theorem need only be proved for the case \( k = 1 \). In fact, if

\[ R(i_1, \ldots, i_k) = X_{i_1} \times \cdots \times X_{i_k}, \]

then condition I_\( k \) states that
\[ (2.1) \quad \lambda^{-1}(i_1) \cdots \lambda^{-1}(i_k) p^{(k)}[R(i_1 \cdots i_k) \cap (T^{(k)})^{-n}R(j_1 \cdots j_k)] > 0 \]

for some \( n > 0 \). Condition II\(_k\) states that

\[ (2.2) \quad \sum_{n=1}^{\infty} [\lambda(0)]^{-k} p^{(k)}[R(0, \cdots, 0) \cap (T^{(k)})^{-n}R(0, \cdots, 0)] = \infty. \]

The \( k \)-dimensional direct product \((X^{(k)}, B^{(k)}, p^{(k)}, T^{(k)})\) of the system \((X, \emptyset, p, T)\) can be regarded as 1-dimensional by relabelling the \( k \)-vector states \((i_1, \cdots, i_k)\) with integers. After relabelling, in view of (2.1) and (2.2) conditions I\(_k\) and II\(_k\) become I\(_1\) and II\(_1\).

If I\(_1\) is not satisfied then for some \((i, j), p\)

\[ (i,j) = \lambda^{-1}(i)p(X_0 \cap T^{-n}X_0) = 0 \quad \text{for all} \quad n > 0 \quad \text{and} \quad T \text{is not ergodic.} \]

If II\(_1\) is not satisfied then

\[ \sum_{n=1}^{\infty} p^n(0, 0) < \infty, \]

the state \( X_0 \) is not recurrent [4], and \( T \) is not ergodic since a wandering set of positive measure exists [1].

Suppose I\(_1\) and II\(_1\) are satisfied, then almost all points of \( X_0 \) return infinitely often to \( X_0 \) under both positive and negative iterations of \( T \) and the smallest invariant set containing \( X_0 \) is essentially the whole space \( X \) (cf. [4, §4]).

The remainder of the proof can be completed by showing that the transformation induced by \( T \) on \( X_0 \) [5], is a Bernoulli transformation. The ergodicity of \( T \) then follows from the ergodicity of the induced transformation [5].

3. Examples. Let \(-1 < \epsilon < 1\), and define

\[ Q = \|q(i, j)\|, \quad i = 0, \pm 1, \pm 2, \cdots \]

where \( q(i, i+1) = (1 - \epsilon/i)/2, \quad q(i, i-1) = (1 + \epsilon/i)/2, \quad i \neq 0, \quad q(0, 1) = q(0, -1) = 1/2, \) and \( q(i, j) = 0 \) if \( j \neq i+1 \) and \( j \neq i-1 \).

Let \( M = \{m(i)\}, \quad i = 0, \pm 1, \cdots, \) where

\[ m(0) = 1, \quad m(i) = m(-i) = \frac{\Gamma(1 + \epsilon)i\Gamma(i - \epsilon)}{\Gamma(1 - \epsilon)\Gamma(i + 1 + \epsilon)}, \quad i > 0. \]

One can easily verify that

\[ MQ = M. \]

Let \( Q^2 = \|q^2(i, j)\| \) and put

\[ P = \|p(i, j)\|, \quad i, j = 0, \pm 2, \pm 4, \cdots, \]
where \( p(i, j) = q^2(i, j) \). Let \( \Lambda = \{ \lambda(i) \} \), where \( \lambda(i) = m(i) \). Then \( \Lambda P = \Lambda \) and \( p(i, j) = 0 \) if and only if \( j \neq i - 2 \), \( j \neq i \), and \( j \neq i + 2 \). 

\( P \) satisfies condition \( I_k \) for every \( k = 1, 2, \ldots \). (No difficulties arise from considering matrices \( P \) defined over the lattice of pairs of even integers.)

Moreover,

\[
\sum \lambda(i) = \infty \quad \text{if } -1 < \varepsilon \leq \frac{1}{2}
\]

since

\[
\lambda(n) \sim \frac{\Gamma(1 + \varepsilon)}{\Gamma(1 - \varepsilon)} n^{-2\varepsilon}.
\]

We shall need the following result of Gillis [2].

**Lemma.** For any \( \theta > 0 \) there exists \( K_1 = K_1(\theta) \) such that for all \( N > 1 \),

\[
K_1^{-1} N^{\varepsilon - 1/2 - \theta} < q^2 N(0, 0) < K_1 N^{\varepsilon - 1/2 + \theta}.
\]

Choose a positive integer \( k \) and \( \eta > 0 \) such that

\[
\frac{1}{k} > \frac{1 + \eta}{1 + k}.
\]

Choose \( \varepsilon \) such that

\[
\frac{1}{2} - \frac{1}{k} < \varepsilon < \frac{1}{2} - \frac{1 + \eta}{1 + k}
\]

and \( \theta > 0 \) such that

\[
\theta < \min \left( \varepsilon - \frac{1}{2} + \frac{1}{k}, \frac{1}{2} - \varepsilon - \frac{1 + \eta}{1 + k} \right);
\]

then

\[
- \frac{1}{k} < \varepsilon - \frac{1}{2} - \theta < \varepsilon - \frac{1}{2} + \theta < -\frac{1 + \eta}{1 + k}.
\]

Consequently, by the lemma, there exists \( K_1 = K_1(\theta) \) such that

\[
K_1 N^{-1/k} < K_1 N^{\varepsilon - 1/2 - \theta} < p^N(0, 0) < K_1 N^{\varepsilon - 1/2 + \theta} < K_1 N^{-(1+\eta)/(1+k)}
\]

i.e.,

\[
(p^N(0, 0))^{b+1} < (K_1)^{b+1} N^{-(1+\eta)}
\]

and
\[(\phi^N(0, 0))^k > (K_1)^k N^{-1}.\]

Hence, by the theorem, the Markov transformation defined by \(P\) has ergodic index \(k\).

Finally, if \(e = 1/2\), then again by the lemma

\[\sum_{n=1}^{\infty} [\phi^n(0, 0)]^k = \infty \text{ for } k = 1, 2, \ldots,\]

and consequently the Markov transformation defined by \(P\) has infinite ergodic index.

REFERENCES

2. J. Gillis, Centrally biased discrete random walk, Quart. J. Math. (2) 7 (1956), 144–152.

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