1. The dual space of a symmetric space. Let $S$ be a symmetric space (that is a Riemannian globally symmetric space), and let $I_0(S)$ denote the largest connected group of isometries of $S$ in the compact open topology. It will always be assumed that $S$ is of the noncompact type, that is $I_0(S)$ is semisimple and has no compact normal subgroup $\neq \{e\}$. Let $l$ denote the rank of $S$; then $S$ contains flat totally geodesic submanifolds of dimension $l$. These will be called planes in $S$.

Let $o$ be any point in $S$, $K$ the isotropy subgroup of $G=I_0(S)$ at $o$ and $\mathfrak{f}_o$ and $\mathfrak{g}_o$ their respective Lie algebras. Let $\mathfrak{g}_o=\mathfrak{f}_o+\mathfrak{p}_o$ be the corresponding Cartan decomposition of $\mathfrak{g}_o$. Let $E$ be any plane in $S$ through $o$, $a_0$ the corresponding maximal abelian subspace of $\mathfrak{p}_o$ and $A$ the subgroup $\exp(a_0)$ of $G$. Let $C$ be any Weyl chamber in $a_0$. Then the dual space of $a_0$ can be ordered by calling a linear function $X$ on $a_0$ positive if $X(\sigma T)>0$ for all $\sigma C$. This ordering gives rise to an Iwasawa decomposition of $G$, $G=KAN$, where $N$ is a connected nilpotent subgroup of $G$. It can for example be described by

$$N = \left\{ z \in G \left| \lim_{t \to \pm \infty} \exp(-tH)z \exp(tH) = e \right. \right\},$$

$H$ being an arbitrary fixed element in $C$. The group $N$ depends on the triple $(o, E, C)$. However, well-known conjugacy theorems show that if $N'$ is the group defined by a different triple $(o', E', C')$ then $N'=gNg^{-1}$ for some $g \in G$.

DEFINITION. A horocycle in $S$ is an orbit of a subgroup of the form $gNg^{-1}$, $g$ being any element in $G$.

Let $t \mapsto \gamma(t)$ ($t$ real) be any geodesic in $S$ and put $T_t=s_{t/2}s_0$ where $s_t$ denotes the geodesic symmetry of $S$ with respect to the point $\gamma(t)$. The elements of the one-parameter subgroup $T_t$ ($t$ real) are called transvections along $\gamma$. Two horocycles $\xi_1, \xi_2$ are called parallel if there exists a geodesic $\gamma$ intersecting $\xi_1$ and $\xi_2$ under a right angle such that $T \cdot \xi_1 = \xi_2$ for a suitable transvection $T$ along $\gamma$. For each fixed $g \in G$, the orbits of the group $gNg^{-1}$ form a parallel family of horocycles.

Let $M$ and $M'$, respectively, denote the centralizer and normalizer of $A$ in $K$. The group $W=M'/M$, which is finite, is called the Weyl group.

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PROPOSITION 1.1. The group $G$ acts transitively on the set of horocycles in $S$. The subgroup of $G$ which maps the horocycle $N \cdot o$ into itself equals $MN$.

Let $\hat{S}$ denote the set of horocycles in $S$. Then we have the natural identifications

$$S = G/K, \quad \hat{S} = G/MN$$

the latter of which turns $\hat{S}$ into a manifold, which we call the dual space of $S$.

PROPOSITION 1.2.
(i) The mapping

$$\phi: (kM, a) \rightarrow kaK$$

is a differentiable mapping of $(K/M) \times A$ onto $S$ and a regular $w$-to-one mapping of $(K/M) \times A'$ onto $S'$.

(ii) The mapping

$$\hat{\phi}: (kM, a) \rightarrow kaMN$$

is a diffeomorphism of $(K/M) \times A$ onto $\hat{S}$.

In statement (i) which is well known, $w$ denotes the order of $W$, $A'$ is the set of regular elements in $A$ and $S'$ is the set of points in $S$ which lie on only one plane through $o$.

PROPOSITION 1.3. The following relations are natural identifications of the double coset spaces on the left:

(i) $K \backslash G / K = A / W$;
(ii) $MN \backslash G / MN = A \times W$.

Statement (i) is again well known; (ii) is a sharpening of the lemma of Bruhat (see [6]) which identifies $MAN \backslash G / MAN$ with $W$.

The proofs of these results use the following lemma.

LEMMA 1.4.
(i) Let $s_0$ denote the geodesic symmetry of $S$ with respect to $o$ and let $\theta$ denote the involution $g \rightarrow s_0 g s_0$ of $G$. Then

$$(N \theta(N)) \cap K = \{e\}.$$ 

(ii) Let $C$ and $C'$ be two Weyl chambers in $a_0$ and $G = KAN$, $G = KAN'$ the corresponding Iwasawa decompositions. Then

$$(NN') \cap (MA) = \{e\}.$$ 

2. Invariant differential operators on the space of horocycles. For any manifold $V$, $C^\infty(V)$ and $C_c^\infty(V)$ shall denote the spaces of $C^\infty$
functions on $V$ (respectively, $C^\infty$ functions on $V$ with compact support). Let $D(S)$ and $D(\hat{S})$, respectively, denote the algebras of all $G$-invariant differential operators on $S$ and $\hat{S}$. Let $S(a_0)$ denote the symmetric algebra over $a_0$ and $J(a_0)$ the set of $W$-invariants in $S(a_0)$. There exists an isomorphism $\Gamma$ of $D(S)$ onto $J(a_0)$ (cf. [7, Theorem 1, p. 260], also [9, p. 432]). To describe $D(\hat{S})$, consider $\hat{S}$ as a fibre bundle with base $K/M$, the projection $\hat{p}: \hat{S} \to K/M$ being the mapping which to each horocycle associates the parallel horocycle through $0$. Since each fibre $F$ can be identified with $A$, each $E_\sigma(S(a_0))$ determines a differential operator $U_F$ on $F$. Denoting by $f|_F$ the restriction of a function $f$ on $\hat{S}$ to $F$ we define an endomorphism $D_U$ on $C^\infty(S)$ by

$$(D_U f)|_F = U_F(f|_F) \quad f \in C^\infty(S),$$

$F$ being any fibre. It is easy to prove that the mapping $U \to D_U$ is a homomorphism of $S(a_0)$ into $D(S)$.

**Theorem 2.1.** The mapping $U \to D_U$ is an isomorphism of $S(a_0)$ onto $D(\hat{S})$. In particular, $D(\hat{S})$ is commutative.

Although $G/MN$ is not in general reductive, $D(\hat{S})$ can be determined from the polynomial invariants for the action of $MN$ on the tangent space to $G/MN$ at $MN$ (cf. [8, Theorem 10]). It is then found that the algebra of these invariants is in a natural way isomorphic to $S(a_0)$, whereupon Theorem 2.1 follows. Let $\hat{f}$ denote the inverse of the mapping $U \to D_U$.

3. The Radon transform. Let $\xi$ be any horocycle in $S$, $ds_\xi$ the volume element on $\xi$. For $f \in C^\infty_c(S)$ put

$$\hat{f}(\xi) = \int f(s)ds_\xi, \quad \xi \in \hat{S}.$$ 

The function $\hat{f}$ will be called the **Radon transform** of $f$.

**Theorem 3.1.** The mapping $f \to \hat{f}$ is a one-to-one linear mapping of $C^\infty_c(S)$ into $C^\infty_c(\hat{S})$.

Now extend $a_0$ to a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$; of the corresponding roots let $P_+$ denote the set of those whose restriction to $a_0$ is positive (in the ordering defined by $C$). Put $\rho = \frac{1}{2} \sum_{\alpha \in P_+} \alpha$ and let $\varphi = \hat{\varphi}$ denote the unique automorphism of $S(a_0)$ given by $\varphi(H = H - \rho(H)$ ($H \in a_0$) (cf. [7, p. 260]).

**Theorem 3.2.** Let $\mathcal{D}(\hat{S})$ be given by

$$\mathcal{D}(\hat{S}) = \{ E \in D(\hat{S}) \mid \varphi(E) \in J(a_0) \},$$
and let $D \rightarrow \mathcal{D}$ denote the isomorphism of $D(S)$ onto $\mathcal{D}(S)$ such that 

$\mathcal{D}(\mathcal{D}(D)) = \Gamma(D)$, \quad $D \in D(S)$.

Then

$(Df)^\ast = \mathcal{D}f$ for $f \in C_c^\infty(S)$.

In view of the duality between points and horocycles there is a natural dual to the transform $f \rightarrow \mathcal{D}f$. This dual transform associates to each function $\psi \in C^\infty(S)$ a function $\mathcal{D}\psi \in C^\infty(S)$ given by

$$\mathcal{D}\psi(p) = \int_{\xi \in \mathcal{P}} \psi(\xi) \, dm(\xi), \quad p \in S,$$

where the integral on the right is the average of $\psi$ over the (compact) set of horocycles passing through $p$. We put

$$I_f = (\mathcal{D}f)^\ast, \quad f \in C^\infty_c(S)$$

and wish to relate $f$ and $I_f$.

**Theorem 3.3.** Suppose the group $G = I_0(S)$ is a complex Lie group. Then

$$\Box I_f = cf, \quad f \in C^\infty_c(S),$$

where $c$ is a constant $\neq 0$ and $\Box$ is a certain operator in $D(S)$, both independent of $f$.

We shall now indicate the definition of $\Box$. Let $J$ denote the complex structure of the Lie algebra $\mathfrak{g}_0$. Then the Cartan subalgebra $\mathfrak{h}_0$ above can be taken as $\mathfrak{a}_0 + J\mathfrak{a}_0$ and can then be considered as a complex Cartan subalgebra of $\mathfrak{g}_0$ (considered as a complex Lie algebra). Let $\Delta'$ denote the corresponding set of nonzero roots and for each $\alpha \in \Delta'$ select $H'_\alpha$ in $\mathfrak{h}_0$ such that $B'(H'_\alpha, H) = \alpha(H)$ ($H \in \mathfrak{h}_0$) where $B'$ denotes the Killing form of the complex algebra $\mathfrak{g}_0$. Then $H'_\alpha \in \mathfrak{a}_0$ and the element $\prod_{\alpha \in \Delta'} H'_\alpha$ in $S(\mathfrak{a}_0)$ is invariant under the Weyl group $W$. Then $\Box$ is the unique element in $D(S)$ such that

$$\Gamma(\Box) = \prod_{\alpha \in \Delta'} H'_\alpha.$$

The proof of Theorem 3.3 is based on Theorem 3 in Harish-Chandra [5] (see also Gelfand-Naïmark [4, p. 156]), together with the Darboux equation for $S$ ([9, p. 442]). In the case when $S$ is the space of positive definite Hermitian $n \times n$ matrices a formula closely related
to (1) was given in Gelfand [1]. Radon's classical problem of representing a function in \( \mathbb{R}^n \) by means of its integrals over hyperplanes was solved by Radon [13] and John [10]. Generalizations to Riemannian manifolds of constant curvature were given by Helgason [8], Semyanistyi [15] and Gelfand-Graev-Vilenkin [3].

4. Applications to invariant differential equations. We shall now indicate how Theorem 3.3 can be used to reduce any \( G \)-invariant differential equation on \( S \) to a differential equation with constant coefficients on a Euclidean space. The procedure is reminiscent of the method of plane waves for solving homogeneous hyperbolic equations with constant coefficients (see John [11]).

**Definition.** A function on \( S \) is called a *plane wave* if there exists a parallel family \( \mathcal{E} \) of horocycles in \( S \) such that (i) \( S = \bigcup_{\xi \in \mathcal{E}} \xi \); (ii) For each \( \xi \in \mathcal{E} \), \( f \) is constant on \( \xi \).

Theorem 3.3 can be interpreted as a decomposition of an arbitrary function \( f \in C_c^\infty(S) \) into plane waves.

Now select \( g \in G \) such that \( \mathcal{E} \) is the family of orbits of the group \( gNg^{-1} \). The manifold \( gNg^{-1} \cdot o \) intersects each horocycle \( \xi \in \mathcal{E} \) orthogonally. A plane wave \( f \) (corresponding to \( \mathcal{E} \)) can be regarded as a function \( f^* \) on the Euclidean space \( \mathcal{A} \). If \( D \in \mathcal{D}(S) \), then \( Df \) is also a plane wave (corresponding to \( \mathcal{E} \)) and \( (Df)^* = D_A f^* \), where \( D_A \) is a differential operator on \( \mathcal{A} \). Using the fact that \( aNa^{-1} \subseteq \mathcal{N} \) for each \( a \in \mathcal{A} \) it is easily proved (cf. [7, Lemma 3, p. 247] or [12, Theorem 1]) that \( D_A \) is invariant under all translations on \( \mathcal{A} \). Thus an invariant differential equation in the space of plane waves (for a fixed \( \mathcal{E} \)) amounts to a differential equation with constant coefficients on the Euclidean space \( \mathcal{A} \). Using Theorem 3.3, and the fact that \( \Box \) commutes elementwise with \( \mathcal{D}(S) \), an invariant differential equation for arbitrary functions on \( S \) can be reduced to a differential equation with constant coefficients (and is thus, in principle, solvable).

**Example: The wave equation on \( S \).** For an illustration of the procedure above we give now an explicit global solution of the wave equation on \( S \) (\( I_0(S) \) assumed complex).

Let \( \Delta \) denote the Laplacian on \( S \) and let \( f \in C_c^\infty(S) \). Consider the differential equation

\[
\Delta u = \frac{\partial^2 u}{\partial t^2}
\]

with initial data

\[
(2) \quad u(p, 0) = 0; \quad \left\{ \frac{\partial}{\partial t} u(p, t) \right\}_{t=0} = f(p) \quad (p \in S).
\]
Let $\Delta_A$ denote the Laplacian on $A$ (in the metric induced by $E$), $\|p\|$ the length of the vector $p$ in §3. Given $a \in A$, let $\log a$ denote the unique element $H \in a_0$ for which $\exp H = a$. For simplicity, let $e^p$ denote the function $a \mapsto e^{\langle \log a \rangle}$ on $A$. Let $\xi$ denote the horocycle $N \cdot o$.

Given $x \in G$, $k \in K$, consider the function

$$F_{k,x}(a) = \int f(xka \cdot s) ds_\xi \quad (a \in A)$$

and the differential equation on $A \times R$,

$$\left( \Delta_A - \|p\|^2 \right) V_{k,x}^t = -\frac{\partial^2}{\partial t^2} V_{k,x}^t,$$

with initial data

$$V_{k,x}^0 = 0; \quad \left\{ \frac{\partial}{\partial t} V_{k,x}^t \right\}_{t=0} = e^p F_{k,x}.$$

Equation (3) is just the equation for damped waves in the Euclidean space $A$ and is explicitly solvable (see e.g. [14, p. 88]). The solution of (1) is now given by

$$u(p, t) = c \Box_p (V(p, t)),$$

where

$$(4) \quad V(xK, t) = \int_K V_{k,x}^t(k) dk.$$ 

Here $dk$ is the normalized Haar measure on $K$ and $c$ is the same constant as in Theorem 3.3. It is not hard to see that the integral in (4) is invariant under each substitution $x \mapsto xu$ ($u \in K$) so the function $V(p, t)$ is indeed well defined.

References


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