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**AN INTEGRATION-BY-PARTS FORMULA**

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In 1914, W. H. Young [4] introduced a modification of the Riemann-Stieltjes integral which, for functions \( F \) and \( G \) defined on the real line with \( G \) of bounded variation on each interval and \( F \) suitably restricted, yields an additive interval function:

\[
(Y) \int_a^b F \cdot dG + (Y) \int_a^c F \cdot dG = (Y) \int_a^c F \cdot dG.
\]

In 1959, T. H. Hildebrandt [1] published a study of a certain linear initial-value problem involving these Young integrals, which extended some of the earlier results of H. S. Wall and of the present author (see [2] for discussion and references). In 1962, there was discovered a connection between the Young integral and the interior integral as introduced by S. Pollard in 1920 [3], *viz.*, the systems

\[
U(x) = U(c) + (Y) \int_c^x U \cdot dH \quad \text{and} \quad V(x) = V(c) + (I) \int_c^x dH \cdot V,
\]

with \( H \) a function from the real line to a complete normed ring, are naturally adjoint to one another [2, p. 326]. Both integrals are to be interpreted as limits in the sense of successive refinements of subdivisions of the interval of integration.

Suppose each of \( F \) and \( G \) is a function from the real line to the complete normed ring \( N \). If each of \( F \) and \( G \) is of bounded variation

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1 Presented to the Society, July 18, 1963.
on the interval \([a, b]\) then each of \((Y) \int_a^b F \cdot dG\) and \((I) \int_a^b dF \cdot G\) is known to exist. Hence, the latter integral exists under the condition that \(F\) is of bounded variation on \([a, b]\) and \(G\) is quasicontinuous, \(i.e.,\) each of the limits \(G(x-)\) and \(G(x+)\) exists for each number \(x\). Here is a new connection between these integrals, which also provides a new existence theorem for the former one.

**Theorem A.** If each of \(F\) and \(G\) is a function from the real line to the complete normed ring \(N\), \(F\) is of bounded variation on the interval \([a, b]\), and \(G\) is quasicontinuous, then

\[
(Y) \int_a^b F \cdot dG + (I) \int_a^b dF \cdot G = F(b)G(b) - F(a)G(a).
\]

**Indication of proof.** If \(a \leq x < y < z \leq b\) then

\[
W = F(x)[G(x) - G(x)] + F(y)[G(y) - G(x)] + F(z)[G(z) - G(y)]
\]

\[
+ [F(z) - F(x)]G(y) - [F(z) - F(x)]G(x)
\]

\[
= [F(x) - F(z)]G(x) + [F(z) - F(y)]G(x) + [F(z) - F(y)]G(y),
\]

so that one has the estimate

\[
|W| \leq 2 \left( \int_x^z |dF| \right) (L.U.B. |G(v) - G(u)| \text{ for } x < u < v < z).
\]

**Addendum.** As has been observed by Randolph Constantine (an oral communication in seminar), the hypotheses on \(F\) and \(G\) in Theorem A can be interchanged. To see this, one first notes the identity

\[
[F(z) - F(x)]G(y)
\]

\[
= F(z)[G(z) - G(x)] - F(x)[G(y) - G(x)] - F(x)[G(y) - G(z)];
\]

next, if \(H\) is a simple step-function and \(\{t_p\}_{0}^{n}\) is an increasing numerical sequence with \(t_0 = a\) and \(t_n = b\),

\[
\left| \sum_{1}^{n} [F(t_{2p}) - F(t_{2p-2})]G(t_{2p-1}) - \sum_{1}^{n} [H(t_{2p}) - H(t_{2p-2})]G(t_{2p-1}) \right|
\]

\[
\leq |F - H|_{[a, b]} \left( |G(a)| + |G(b)| + \int_a^b |dG| \right),
\]

where \(|F - H|_{[a, b]} = L.U.B. |F(u) - H(u)| \text{ for } u \text{ in } [a, b]|\), and also

\[
|Y| \int_a^b F \cdot dG - (Y) \int_a^b H \cdot dG| \leq |F - H|_{[a, b]} \left( \int_a^b |dG| \right).
\]
Thus, an argument is easily made to establish the following somewhat stronger theorem.

**Theorem B.** If each of $F$ and $G$ is a quasicontinuous function from the real line to the complete normed ring $N$, and one of $F$ and $G$ is of bounded variation on the interval $[a, b]$, then

$$(Y) \int_a^b F \cdot dG + (I) \int_a^b dF \cdot G = F(b)G(b) - F(a)G(a).$$

**Remark.** The reader is invited to contrast this formula with the corresponding result involving Young integrals alone (or interior integrals alone), as obtained by Hildebrandt [1, p. 355] for the case that both $F$ and $G$ are of bounded variation. For this case, there is a more general result available, as indicated in the following theorem.

**Theorem C.** If Axioms I and II [2, p. 321] hold, each of $F$ and $G$ is a function from the interval $[a, b]$ to $N$, and $dG(x, z) = K_1(x, z)$ and $dF(x, z) = K_2(x, z)$ for $a \leq x < z \leq b$, then

$$K_1[F](a, b) + K_2[G](a, b) = F(b)G(b) + \sum_{a < x < b} \{dF \cdot K_1[1_x] + K_2[1_x] \cdot dG - dF \cdot dG\}(z-, z) - F(a)G(a) - \sum_{a < z < b} \{dF \cdot K_1[1_z] + K_2[1_z] \cdot dG - dF \cdot dG\}(x, x+).$$

**Remark.** After obtaining the preceding results, the author learns (July 27, 1963) that Theorem B has been discovered by T. H. Hildebrandt (on May 28, 1963) for **numerical valued functions** $F$ and $G$: that priority of discovery is hereby cordially acknowledged to Professor Hildebrandt.

**Bibliography**


**The University of North Carolina at Chapel Hill**