ON DIFFERENTIABLE IMBEDDINGS OF SIMPLY-CONNECTED MANIFOLDS

BY J. LEVINE

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1. Introduction. We will be concerned with the problem of imbedding (differentiably) a closed simply-connected \( n \)-manifold \( M \) in the \( m \)-sphere \( S^m \). According to [3], this problem depends only upon the homotopy type of \( M \), in a “stable” range of dimensions. We obtain an explicit equivalent homotopy problem.

We also consider the problem of determining whether two imbeddings of \( M \) in \( S^m \) are isotopic (see [3] for basic definitions). A “homotopy condition” for deciding this question will also be obtained, again in a “stable” range of dimensions.

All manifolds, imbeddings and isotopies are to be differentiable. If \( M, V \) are manifolds with boundary and \( f \) is an imbedding of \( M \) in \( V \), it will always be understood that \( f(M) \cap \partial V = f(\partial M) \) and the intersection is transverse.

2. Imbedding theorem. \( M \) will, hereafter, denote a closed simply-connected \( n \)-manifold, \( n > 4 \). Suppose \( f \) imbeds \( M \) in \( S^m \); then we can define the normal plane bundle \( v_f \) and, by a construction of Thorn [10], an element \( \alpha_f \in \pi_m(T(v_f)) \), where \( T(v_f) \) is the Thom space (see [10]) of \( v_f \). We call the pair \( (v_f, \alpha_f) \) the normal invariants of \( f \). The existence of an imbedding, in particular, implies the existence of an \( (m-n) \)-plane bundle \( \xi \) whose Thom space is reducible in the sense of [1]. It follows from [1] that this property of \( M \) is a homotopy invariant and such a bundle \( \xi \) must be, a priori, stably fiber homotopy equivalent to the stable normal bundle of \( M \).

Let \( M_0 \) denote the complement of an open disk in \( M \).

**Theorem 1.** Suppose \( 2m \geq 3(n+1) \) and \( \xi \) is an \((m-n)\)-plane bundle over \( M \) stably equivalent to the stable normal bundle of \( M \), such that \( T(\xi) \) is reducible. Then there is an imbedding \( f \) of \( M \) in \( S^m \) such that \( v_f \) is fiber homotopy equivalent to \( \xi \):

(a) Over \( M \) if \( n = 6, 14 \) or \( n \neq 2 \mod 4 \).

(b) Over \( M_0 \) if \( n = 2 \mod 4 \).

It is to be expected that the conclusion of (a) is valid for all \( n \). The difficulty in the proof arises from the lack of a satisfactory general definition of the Arf invariant (see [7]). In certain special cases, e.g., if \( M \) is a \( \pi \)-manifold or \( \pi_i(M) = 0 \) for \( 2i < n \), we can obtain the conclusion of (a).
3. Isotopy theorems. Suppose \( f, g \) are isotopic imbeddings of \( M \) in \( S^m \). It is easy to show that there is a bundle map \( \phi: \nu_f \to \nu_g \) (note that the terminology implies that \( \phi \) covers the identity map of \( M \)) such that \( \phi_* (\alpha_f) = \alpha_g \). We say \( \phi \) induces an equivalence between the normal invariants of \( f \) and \( g \).

**Theorem 2.** Suppose \( 2m > 3(n+1) \). Then two imbeddings of \( M \) in \( S^m \) are isotopic if and only if they have equivalent normal invariants.

Theorems 1 and 2 represent alternatives to the classification theorems of [5]. Note that the normal bundle plays a more prominent role here; in particular, Theorem 1 gives us information on the possible normal bundles of imbeddings.

The situation is more complicated in the borderline case \( 2m = 3(n+1) \). For \( n = 4k-1 \), \( m = 6k \), we obtain a generalization of the main result of [4]. Let \( (\xi, \alpha) \) be the normal invariants of an imbedding of \( M \) in \( S^m \); we will define a cyclic group \( Z(\xi, \alpha) \). If \( f, g \) are imbeddings of \( M \) in \( S^m \) whose normal invariants are equivalent to \( (\xi, \alpha) \) we define a further invariant \( L(f, g) \in Z(\xi, \alpha) \).

For the following theorem we must impose an additional restriction upon \( M \):

(*) If \( H \) is a homotopy \( n \)-sphere which bounds a \( \pi \)-manifold and the connected sum \( M \# H \) is diffeomorphic to \( M \), then \( H \) is diffeomorphic to \( S^n \).

**Theorem 3.** Suppose \( n = 4k - 1, m = 6k \).

(a) If \( f, g \) are imbeddings of \( M \) in \( S^m \) with equivalent normal invariants, then \( f \) and \( g \) are isotopic if and only if \( L(f, g) = 0 \).

(b) If \( f \) is an imbedding with normal invariants \( (\xi, \alpha) \) and \( L \in Z(\xi, \alpha) \), then there exists an imbedding \( g \) whose normal invariants are equivalent to \( (\xi, \alpha) \) such that \( L(f, g) = L \).

Thus \( L(f, g) \) plays the role of a difference cochain in obstruction theory. One may conjecture on the existence of a higher obstruction theory for imbeddings with equivalent normal invariants in the "non-stable" range of dimensions.

4. Discussion of proofs. We use a nonstable version of the procedures introduced in [2; 9] (see [8] for details). To prove Theorem 1 we construct a submanifold \( N \) of \( S^m \), and a map \( h: N \to M \) of degree \(+1\) such that \( h^* \xi \) is the normal bundle of \( N \) in \( S^m \). By a suitable generalization of the techniques in [4, §3], we can perform spherical modifications on the pair \((S^m, N)\), at each stage defining a new map \( h \) so that \( h^* \xi \) is still the normal bundle. We must use the restriction
on codimension here. Following [2; 8] we can eventually make \( h \) a homotopy equivalence, if \( n = 6, 14 \) or \( n \equiv 2 \mod 4 \). If \( n \equiv 2 \mod 4 \), we must replace \( S^m \) by the \( m \)-disk \( D^m \) and let \( N \) be a bounded manifold imbedded in \( D^m \), with \( \partial N \) a homotopy sphere and \( h: N \rightarrow M_0 \) such that \( h^*(\xi|_{M_0}) \) is the normal bundle to \( N \) in \( D^m \). Now we can perform spherical modifications as above to make \( h \) a homotopy equivalence.

Using the results of [3; 6] one can deform a homotopy inverse of \( h \) into an imbedding of \( M \) (or \( M_0 \)) into \( S^m \) (or \( D^m \)) with a normal bundle fibre homotopy equivalent to \( \xi \) (or \( \xi|_{M_0} \)). If \( n \equiv 2 \mod 4 \), the imbedding of \( M_0 \) into \( D^m \) can be extended to \( M \) into \( S^m \).

Theorem 2 is approached by a similar combination of the methods of [9] (see [8]) and the techniques of [4]. At one point we need the following result, which follows easily, in this range of dimensions, from the results of [3]. Let \( H \) be a homotopy \( n \)-sphere and \( g: M \rightarrow M \# H \) a diffeomorphism homotopic to the standard homeomorphism. Let \( f \) be an imbedding of \( M \) in \( S^m \), \( i \) an imbedding of \( H \) in \( S^m \) and \( f' \) the imbedding (unique up to isotopy) of \( M \# H \) in \( S^m \) induced by \( f \) and \( i \). Then \( f' \cdot g \) is isotopic to \( f \).

The proof of Theorem 3 proceeds as that of Theorem 2 up to a point. In order to complete the necessary spherical modifications we must consider a linking number invariant (an integer) similar to that defined in [4]. After reducing to a quotient group, \( Z(\xi, \alpha) \), we obtain \( L(f, g) \) which depends only on \( f \) and \( g \). Now, the verification of (a) is not unlike the arguments in [4, §3]; (b) is proved by adjoining to \( f \) one of the knotted spheres constructed in [4].

5. More general results. Given a closed \( m \)-manifold \( V \) and \( v \in H_n(V) \), we may ask whether \( v \) can be realized by an imbedding \( f \) of \( M \). If so, we can define the normal bundle \( v_f \) and \( \alpha_f \in \pi(V, T(v_f)) \) (= homotopy classes of maps \( V \rightarrow T(v_f) \)), using the procedures of [10], such that \( \alpha^*_f(u(v_f)) = \text{dual of } v \) (if \( \xi \) is a \( k \)-plane bundle over \( M \), \( u(\xi) \in H^k(T(\xi)) \) is the usual generator). Also \( f^*\tau_V = \tau_M + v_f \), where \( \tau_M \) denotes the tangent bundle of \( M \). In particular, if \( n = 4k \) and \( \hat{p}_i(V) = 0 \) for \( 0 < i < k \):

\[
\text{index } M = L_k(\bar{p}_1(\xi), \ldots, \bar{p}_{k-1}(\xi), \bar{p}_k(\xi) + \langle \hat{p}_k(V), v \rangle M)
\]

where \( \bar{p}_i(\xi) \) is the dual Pontryagin class of \( \xi \), \( L_k \) is the Hirzebruch polynomial (see [7]) and \( \xi = v_{\mathcal{F}} \).

**Theorem 4.** Let \( M, V, v \) be as above and \( 2m \geq 3(n+1) \). Assume \( \pi_i(V) = 0 \) for \( 2i \leq n \). Suppose \( \xi \) is an \( (m-n) \)-plane bundle over \( M \) satisfying (†) if \( n = 4k \) and there exists \( \alpha \in \pi(V, T(\xi)) \) such that \( \alpha^*(u(\xi)) \)
dual of $v$. Then there is an imbedding $f$ of $M$ in $V$, representing $v$, such that $v_f$ is fiber homotopy equivalent to $\xi$:

(a) Over $M$ if $n = 6$, 14 or $n \equiv 2 \mod 4$.
(b) Over $M_0$ if $n \equiv 2 \mod 4$.

The proof is similar to that of Theorem 1. There is also an isotopy theorem.

**BIBLIOGRAPHY**

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