ON THE STRUCTURE OF SEMI-NORMAL OPERATORS

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1. Preliminaries. Only bounded operators on a Hilbert space $\mathcal{H}$ of elements $x$ will be considered. If $A$ is self-adjoint with the spectral resolution

$A = \int \lambda dE(\lambda),$

and if $\mathfrak{H}_a = \mathfrak{H}_a(A)$ denotes the set of elements $x$ for which $\|E(\lambda)x\|^2$ is an absolutely continuous function of $\lambda$, then $\mathfrak{H}_a$ is a subspace; cf. [2, p. 240], [3, p. 436] and [6, p. 104]. If $\mathfrak{H} = \mathfrak{H}_a$, then $A$ is called absolutely continuous. The one-dimensional Lebesgue measure of the spectrum of a self-adjoint operator $A$ will be denoted by $\text{meas} \, \text{sp}(A)$.

An operator $T$ on $\mathfrak{H}$ is called semi-normal if

$TT^* - T^*T = D \geq 0$ or $D \leq 0.$

There will be proved the following result concerning such an operator.

2. Theorem. If $T$ satisfies (2) and if $\mathcal{M} = \mathcal{M}_T$ is the smallest subspace of $\mathfrak{H}$ reducing $T$ and containing the range of $D$, then

$T + T^*$ is absolutely continuous on $\mathcal{M},$

and, if $\mathcal{M}^\perp$ denotes the orthogonal complement of $\mathcal{M}$ (so that $\mathcal{M}^\perp$ also reduces $T$), then

$T$ is normal on $\mathcal{M}^\perp.$

In addition,

$2\pi \|D\| \leq \|T - T^*\| \, \text{meas} \, \text{sp}(T + T^*),$

and the inequality (5) is optimal in the sense that there exist examples with $D \neq 0$ for which (5) becomes an equality.

As a consequence, if $T$ is semi-normal but not normal, then $\mathfrak{H}_a(T + T^*) \neq 0$, a result which can also be concluded from [4, Corollary 3, p. 1029], where the symbol "<' should be replaced by "$\neq". (This same Corollary, incidentally, also implies the result proved by Andô [1] that a completely continuous semi-normal operator $T$ must be normal. In fact, if $T$ is completely continous, so also are $T^*$ and $T + T^*$. But the spectrum of $T + T^*$ clearly must be of measure zero.)

If $\theta$ is real and $T(\theta) = e^{i\theta}T$, then (2) is unchanged if $T$ is replaced by $T(\theta)$. Also, it is clear that the set $\mathcal{M}_{T(\theta)}$ is independent of $\theta$. It follows that (3), (4) and (5) remain valid if, in each instance, $T$ is

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replaced by \( T(\theta) \). In particular then, relations (3) and (5) become assertions concerning the absolute continuity and spectra of both the real and the imaginary parts of a semi-normal operator \( T \).

The proof of the Theorem will depend upon results proved in [5] and which will be stated here, in a form convenient for application, as a

**Lemma.** Let \( H \) and \( J \) be self-adjoint operators and suppose that
\[
(6) \quad HJ - JH = iC, \text{ where } C \supseteq 0 \text{ or } C \subseteq 0.
\]
Then,
\[
(7) \quad \mathcal{Q} \subseteq \mathcal{S}_a(H),
\]
where \( \mathcal{Q} \) denotes the smallest subspace reducing both \( H \) and \( J \) and also containing the range of \( C \). Furthermore,
\[
(8) \quad \pi ||C|| \leq ||J|| \text{ meas sp}(H).
\]

It is clear from the symmetry of the condition (6) that (7) and (8) remain true if \( H \) and \( J \) are interchanged.

3. **Proof of the Theorem.** Let \( T \) be represented as
\[
(9) \quad T = H + iJ, \text{ where } H = (T + T^*)/2 \text{ and } J = (T - T^*)/2i,
\]
so that (2) and (6) are equivalent by virtue of (9) and
\[
(10) \quad D = 2C.
\]
It is clear that the space \( \mathcal{Q} \) of the Lemma must then coincide with the space \( \mathcal{M} \) of the Theorem. Relations (3) and (5) now follow respectively from (7) and (8), while relation (4) is a consequence of the fact that \( \mathcal{M}^\perp \) is contained in the null space of \( D \). An example involving finite interval Hilbert transforms was given in [5] for which the hypothesis of the Lemma is fulfilled and for which (8) becomes an equality (with \( C \neq 0 \)). This result in turn yields, by virtue of (9) and (10), an example in which equality holds in (5) and \( D \neq 0 \).

**References**

5. ———, *Commutators, absolutely continuous spectra, and singular integral operators* (to appear).

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