1. Introduction. Brown [1] has shown that an $S^{n-1}$ embedded in a locally flat manner in $S^n$ is flat and hence tame in $S^n$. Bing [2] and Moise [3] have shown that locally tame subsets of 3-manifolds are tame. However, in the general case, it is not known whether a manifold $N$ embedded in a locally flat manner in a triangulated manifold $M$ or a polyhedron $P$ embedded in a locally tame manner in a triangulated manifold $M$ are tame in $M$. Partial solutions to both of these problems have been obtained by the author and will be stated in §3 of this paper. I have been informed by R. H. Bing that Herman Gluck has obtained similar results.

2. Definitions and notations. Let $N^k$ be a combinatorial $k$-manifold. Then $(N^k)^r$ will denote the $r$th barycentric subdivision of $N^k$. If $\alpha$ is a $k$-simplex of $(N^k)^r$ and $\alpha''$ is the union of all simplices of $(N^k)^{r+2}$ contained in $\alpha$, then $C_{\alpha}$ will denote the closed simplicial neighborhood of $|\alpha''|$, the polyhedron of $\alpha''$, in $(N^k)^{r+2}$. That is $C_{\alpha}$ is the union of all closed simplices in $(N^k)^{r+2}$ that meet $|\alpha''|$. Since $\alpha''$ is collapsible, $C_{\alpha}$ is a combinatorial $k$-ball [4].

The statement that $f$ is a locally flat embedding of a $k$-manifold $N^k$ in an $n$-manifold $N^n$, means that each point of $f(N^k)$ has a neighborhood $U$ in $N^n$ such that the pair $(U, U\cap f(N^k))$ is homeomorphic to the pair $(R^n, R^k)$.

Two definitions of locally tame will now be given.

**DEFINITION 1.** Let $N$ be a manifold topologically embedded in a triangulated manifold $M$. $N$ is locally tame if for each point $p$ of $N$, there exists a neighborhood $U$ of $p$ in $M$ and a homeomorphism $h$ of $U$ into $M$, such that $h[\text{Cl}(U\cap N)]$ is a polyhedron in $M$.

**DEFINITION 2.** Let $P$ be a polyhedron topologically embedded in a triangulated manifold $M$. $P$ is locally tame if for each point $p$ of $P$, there exists a neighborhood $U$ of $p$ in $M$ and a homeomorphism $h$ of $U$ into $M$, such that $h[\text{Cl}(U\cap P)]$ is piecewise linear with respect to a fixed triangulation $T$ of $P$.

Let $K$ be a complex topologically embedded by $f$ in a triangulated $n$-manifold $N^n$ and let $\epsilon > 0$. Suppose there exists an $\epsilon$-homeomorphism $h$ of $N^n$ onto itself such that if $U_\epsilon(f(K))$ denotes the set of points in $N^n$ whose distance from $f(K)$ is less than $\epsilon$, then
(i) \( h|N^n - U_*(f(K)) = 1 \),
(ii) \( hf: K \to N^n \) is a piecewise linear embedding.
Then \( f(K) \) will be said to be \( \varepsilon \)-tame in \( N^n \).


**Theorem 1.** Let \( f \) be a locally flat embedding of a closed combinatorial \( k \)-manifold \( N^k \) in a closed combinatorial \( n \)-manifold \( N^n \), \( 2k + 2 \leq n \) and \( \varepsilon > 0 \). Then \( f(N^k) \) is \( \varepsilon \)-tame in \( N^n \).

**Theorem 2.** Let \( f_1 \) and \( f_2 \) be locally flat (locally tame) embeddings of a closed combinatorial \( k \)-manifold \( N^k \) (finite \( k \)-polyhedron \( P^k \)) in \( S^n \) and \( 2k + 2 \leq n \). Then there exists a homeomorphism \( h \) of \( S^n \) onto itself such that \( hf_1 = f_2 \).

**Theorem 3.** Let \( f \) be a locally flat embedding of a \( k \)-manifold \( N^k \) in a combinatorial \( n \)-manifold \( N^n \) and \( 2k + 2 \leq n \). Then \( f(N^k) \) is locally tame (Definition 1).

**Theorem 4.** Let \( f \) be a locally tame (Definition 2) embedding of a possibly infinite \( k \)-polyhedron \( P^k \) as a closed subset of the interior of a combinatorial \( n \)-manifold \( N^n \), \( 2k + 2 \leq n \) and \( \varepsilon > 0 \). Then \( f(P^k) \) is \( \varepsilon \)-tame in \( N^n \).

4. Reference theorems.

**Homma's Theorem [5].** Let \( M^n \), \( \hat{M}^n \) and \( \hat{P}^k \) be two finite combinatorial \( n \)-manifolds and a finite polyhedron such that \( \hat{M}^n \) is topologically embedded in \( M^n \), \( \hat{P}^k \) is piecewise linearly embedded in \( \text{Int}(M^n) \) and \( 2k + 2 \leq n \). Then for \( \varepsilon > 0 \), \( \hat{P}^k \) is \( \varepsilon \)-tame in \( M^n \).

**Gluck's Modification of Homma's Theorem [6].** Let the following be given:
(i) \( M^n \), a possibly noncompact combinatorial \( n \)-manifold;
(ii) \( \hat{M}^n \), a possibly noncompact combinatorial \( n \)-manifold, topologically embedded in \( M^n \);
(iii) \( \hat{P}^k \), a possibly infinite polyhedron, piecewise linearly embedded as a closed subset of \( \text{Int}(\hat{M}^n) \);
(iv) \( \hat{L} \), a subpolyhedron of \( \hat{P}^k \) such that \( \text{Cl}(\hat{P}^k - \hat{L}) \) is a finite polyhedron, and such that \( \hat{L} \) is piecewise linearly embedded in \( M^n \) as well as in \( \hat{M}^n \).

If \( 2k + 2 \leq n \), then for any \( \varepsilon > 0 \), there is an \( \varepsilon \)-homeomorphism \( F \) of \( M^n \) onto \( M^n \) such that under \( F \), \( \hat{P}^k - \hat{L} \) is \( \varepsilon \)-tame in \( M^n \) and \( F|\hat{L} = 1 \).

5. Partial proofs of results.

**Lemma 1.** Suppose the following are given:
(i) The hypotheses of Theorem 1 are satisfied.

(ii) \( \{ (U_i, U_i \cap f(N^k)) \text{, } i = 1, \ldots, q \} \) is a finite open cover of \( f(N^k) \) obtained by applying the definition of locally flat.

(iii) \( \epsilon > 0 \).

Then there exists an integer \( r \) such that if \( \alpha \) is a \( k \)-simplex of \( (N^k)_r \) and if \( C_{f(\alpha)} = f(C_\alpha) \),

(a) \( f(\alpha) \subseteq C_{f(\alpha)} \subseteq U_j \cap f(N^k) \) for some \( j \).

(b) \( C_{f(\alpha)} \) is \( \epsilon \)-tame in \( N^k \).

Conclusion (a) is obvious since every open cover of a compact metric space has a Lebesgue number and the limit of the mesh of \( f(N^k)_i \) as \( i \) approaches infinity is zero.

Let \( r \) and \( j \) be integers such that conclusion (a) is true. Let \( h_j \) be the homeomorphism of \( (U_j, U_j \cap f(N^k)) \) onto \( (R^n, R^k) \). Since \( C_\alpha \) is a bicollared \([1] \) \( k-1 \) sphere in \( N^k \), \( h_j(f(C_\alpha)) \) is a bicollared \( k-1 \) sphere in \( R^k \). Hence \( h_j(f(C_\alpha)) \) is a tame \( k \)-cell in \( R^k \) and therefore \( U_j \) can be triangulated as a combinatorial \( n \)-manifold in such a way that \( f: C_\alpha \to U_j \) is a piecewise linear embedding.

We now apply Homma's theorem. Let \( M^n = N^n \) be a closed regular neighborhood of \( C_{f(\alpha)} \) in \( U_j \) and \( \hat{P}^n = C_{f(\alpha)} \). Homma's theorem asserts that \( C_{f(\alpha)} \) is \( \epsilon \)-tame in \( N^n \).

PROOF OF THEOREM 1. Let \( r \) be an integer such that if \( \alpha \) is a \( k \)-simplex of \( (N^k)_r \), Lemma 1 is valid. Let \( A_i \) denote the proposition that if \( K_i \) is a connected homogeneous \( k \)-subcomplex of \( (N^k)_r \) containing \( i \) \( k \)-simplexes, then \( f(K_i) \) is \( \epsilon \)-tame in \( N^n \) for each \( \epsilon > 0 \). It suffices to show that \( A_i \) is true for each positive integer \( i \).

\( A_1 \) is true by Lemma 1. Suppose \( A_i \) is true for \( 1 \leq i \leq j \). Let \( K_{j+1} \) be a connected homogeneous \( k \)-subcomplex of \( (N^k)_r \) containing \( j+1 \) \( k \)-simplexes. Then \( K_{j+1} = K_j \cup \alpha \), where \( K_j \) is a connected homogeneous \( k \)-subcomplex of \( (N^k)_r \) containing \( j \) \( k \)-simplexes and \( \alpha \) is a \( k \)-simplex of \( (N^k)_r \). Let \( \epsilon > 0 \) and \( \epsilon' = \epsilon/2 \), then by assumption, \( f(K_i) \) is \( \epsilon' \)-tame in \( N^n \) and by Lemma 1, \( C_{f(\alpha)} \) is \( \epsilon' \)-tame in \( N^n \).

Let \( h_k \) and \( h_\alpha \) be the \( \epsilon' \)-homeomorphisms for \( f(K_j) \) and \( C_{f(\alpha)} \) respectively such that they are \( \epsilon' \)-tame in \( N^n \). Let \( U_\alpha \) be an open ball neighborhood of \( h_\alpha(C_{f(\alpha)}) \) in \( N^n \), and \( W_\alpha = h_\alpha^{-1}(U_\alpha) \).

We will complete the proof of \( A_{j+1} \) by applying Gluck's modification of Homma's theorem. Let \( M^n = h_k(w_\alpha) \) triangulated as an open subset of \( N^n \), \( \hat{M}^n = h_k(W_\alpha) \) triangulated as a combinatorial \( n \)-manifold such that \( h_k f: C_\alpha \to h_k(W_\alpha) \) is a piecewise linear embedding. Take \( \hat{P}^n = h_k[C_{f(\alpha)} \cap f(K_j)] \cup h_k(f(\alpha)) \) and \( \hat{L} = h_k[C_{f(\alpha)} \cap f(K_j)] \). By choice of \( h_k \), \( \hat{L} \) is piecewise linearly embedded in both \( M^n \) and \( \hat{M}^n \). Let \( \epsilon'' \) be picked such that \( 0 < \epsilon'' < \epsilon' \) and such that \( [U_{\epsilon''}, (h_k(f(\alpha)))] \cap h_k(f(K_j)) \subseteq \hat{L} \).
and \( \text{Cl}[U_{b'}(f(\alpha))) \subset h_k(W_a) \). The hypotheses of Gluck's theorem are satisfied, hence there exists an \( \epsilon'' \)-homeomorphism \( g \) of \( M^n \) onto itself such that \( \hat{P}^k - \hat{L} \) is \( \epsilon'' \)-tame in \( M^n \) under \( g \) and \( g|\hat{L} = 1 \). \( g \), which is the identity on \( h_k[f(K_j) \cap W_a] \) and near the boundary of \( h_k(W_a) \), may be extended via the identity to an \( \epsilon'' \)-homeomorphism \( \tilde{g} \) of \( N^n \) onto itself.

Then \( F = \tilde{g}h_k \) is an \( \epsilon \)-homeomorphism of \( N^n \) onto itself, such that under \( F, f(K_{j+1}) \) is \( \epsilon \)-tame in \( N^n \). Thus \( A_{j+1} \) is true and by induction the theorem is proved.

Theorems 1 and 4 reduce the proof of Theorem 2 to the piecewise linear case which has already been handled in [7].

The proof of Theorem 3 is an easy application of Homma's theorem. The following lemma also follows from Homma's theorem.

**Lemma 2.** Suppose the following are given:

(i) The hypotheses of Theorem 4 are satisfied except \( P^k \) is finite.
(ii) \( \{(U_\lambda, U_\lambda \cap f(P^k), \lambda = 1, \cdots, q\} \) is a finite open cover of \( f(P^k) \) obtained by applying Definition 2 of locally tame.
(iii) \( \epsilon > 0 \).

Then there exists a triangulation of \( f(P^k) \) such that the closed simplicial neighborhood of any simplex in this triangulation of \( f(P^k) \) is contained in \( U_\lambda \cap f(P^k) \) for some \( j \) and is \( \epsilon \)-tame in \( N^n \).

Lemma 2, together with Gluck's modification of Homma's theorem are sufficient to prove Theorem 4.

Actually, Lemma 1 shows that locally flat closed combinatorial manifolds with the correct codimension are locally tame according to Definition 2. This, together with Theorem 4, would yield Theorem 1 as a corollary.

**References**


Florida State University