REPRESENTING MEASURES FOR POINTS IN
A UNIFORM ALGEBRA

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Recent years have seen much effort put into attempts to develop
an "abstract" function theory. In the complex domain, this has led
to the study of the structure of uniform algebras. A uniform algebra
\( \mathfrak{A} \) is a family of continuous complex-valued functions on a compact
Hausdorff space \( X \), which contains the function 1, which is closed
with respect to the algebraic operations of addition and multiplica­
tion by complex scalars, which is topologically closed in the uniform
norm, and which distinguishes points of \( X \). Succinctly, \( \mathfrak{A} \) is a closed
separating unitary subalgebra of the Banach algebra \( C(X) \) of all
continuous complex-valued functions on \( X \). Standard examples are
obtained by taking \( X \) to be a compact subset of complex Euclidean
space \( \mathbb{C}^n \) and \( \mathfrak{A} \) to be the closed subalgebra of \( C(X) \) generated by the
constants and the coordinate functions on \( \mathbb{C}^n \).

Some of the most suggestive ideas in the theory of uniform alge­
bras come from a paper of Gleason [1]. He calls elements \( x \) and \( y \) in
\( X \) equivalent if

\[
d(x, y) = \sup \{ |f(x) - f(y)| : f \in \mathfrak{A}, \|f\| \leq 1 \} < 2.
\]

This is an equivalence relation on \( X \). The equivalence classes are
called parts or Gleason parts. The parts have various other character­
izations, obtainable from the elementary conformal geometry of the
disc \( D = \{ z : |z| \leq 1 \} \). We shall need to know, in particular, that if
there exists a sequence \( \{ h_n \} \) of elements of \( \mathfrak{A} \), \( \|h_n\| \leq 1 \), \( |h_n(x)| \to 1 \) as
\( n \to \infty \), then \( |h_n(y)| \to 1 \) as \( n \to \infty \) if \( x \) and \( y \) are in the same part.

Gleason showed, in a special case, that parts can be given a certain
characterization in terms of representing measures. A representing
measure \( \mu_x \) for a point \( x \) in \( X \) is a non-negative Baire measure on \( X \)
such that \( \int f d\mu_x = f(x) \) for all \( f \) in \( \mathfrak{A} \). The following theorem generalizes
Gleason's result to the general situation.

**Theorem 1.** If \( x \) and \( y \) are in the same part, there exists \( c > 0 \) and
representing measures \( \mu_x \) for \( x \) and \( \mu_y \) for \( y \) such that \( c \mu_x \leq \mu_y \) and
\( c \mu_y \leq \mu_x \).

**Proof.** Let \( C_r(X) \) be the Banach space of all continuous real-
valued functions on \( X \). Let \( R \) be the set of all \( f \) in \( C_r(X) \) such that
\( f + ig \in \mathfrak{A} \) for some \( g \) in \( C_r(X) \). Let \( c \) be a constant, \( 0 < c < 1 \), such that
there exists $f$ in $\mathbb{R}$ with $f \geq 0$, $f(y) = 1$, and $f(x) < c$. Then with $g$ as above we have

$$h = e^{-f+io} \in \mathcal{Y},$$

$||h|| \leq 1$, $|h(y)| = e^{-1}$, $|h(x)| \geq e^{-c}$. Now if $c$ can be chosen arbitrarily near to 0 we see from the above that $x$ and $y$ would not be in the same part. Thus there exists $c < 1$ such that for all $f$ in $\mathbb{R}$, $f \geq 0$, we have $f(x) \geq cf(y)$. For reasons of symmetry we may choose the constant $c$ so that in addition $f(y) \geq cf(x)$, $f$ in $\mathbb{R}$, $f \geq 0$. By a standard theorem, it follows that there exists a positive measure $\alpha$ on $X$ such that $f(x) - cf(y) = \int f d\alpha$ for all $f$ in $\mathbb{R}$ and thus for all $f$ in $\mathcal{Y}$. Similarly, there exists a positive measure $\beta$ on $X$ such that $\int f d\beta = f(y) - cf(x)$ for all $f$ in $\mathcal{Y}$. Thus

$$f(y) = cf(x) + \int f d\beta = c^2 f(y) + c \int f d\alpha + \int f d\beta,$$

so

$$f(y) = (1 - c^2)^{-1} \left[ c \int f d\alpha + \int f d\beta \right].$$

Similarly,

$$f(x) = (1 - c^2)^{-1} \left[ c \int f d\beta + \int f d\alpha \right].$$

Thus the theorem holds with

$$\mu_x = (1 - c^2)^{-1}[cd\beta + d\alpha],$$

$$\mu_y = (1 - c^2)^{-1}[cd\alpha + d\beta].$$

COROLLARY. If $\mu_x'$ is any representing measure for $x$, there exists a representing measure $\mu_y'$ for $y$ with $c\mu_x' \leq \mu_y'$.

PROOF. Take $\mu_x$ and $\mu_y$ as above, and write $\mu_y' = (\mu_y - c\mu_x) + c\mu_x'$. We remark that a lower bound for $c$ could be explicitly computed in terms of $d(x, y)$.

REFERENCES


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