THE DIMENSION OF THE SUPPORT OF A RANDOM DISTRIBUTION FUNCTION

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In their paper Random distribution functions (Bull. Amer. Math. Soc. 69 (1963), 548–551) L. E. Dubins and D. A. Freedman defined a random distribution function \( F \) associated with a probability measure \( \mu \) on the unit square \( S \) whose values are distribution functions on \([0, 1]\). To choose a value \( F_w \) of \( F \) they proceed as follows: Points \( P(n, j) \) of \( S \) are defined inductively for all \( n \) and \( j = 0, \ldots, 2^n \) by setting \( P(0, 0) = (0, 0) \), \( P(0, 1) = (1, 1) \), \( P(n+1, 2j) = P(n, j) \) and \( P(n+1, 2j+1) \) equal to the image under the unique affine transformation carrying \( S \) onto the rectangle \( R(P(n, j), P(n, j+1)) \) formed by the vertical and horizontal lines through \( P(n, j) \) and \( P(n, j+1) \) of a point \( P^*(n+1, 2j+1) = (x^*(n, 2j+1), y^*(n, 2j+1)) \) chosen according to the distribution \( \mu \) independently of the previous choices. They showed that \( \cap_{n=1}^{\infty} \cup_{j=0}^{2^n} R(P(n, j), P(n, j+1)) \) is the graph of a continuous monotone function \( F_w(x) \) increasing from 0 to 1 on \([0, 1]\), that is, a distribution function defining a measure \( \mathcal{P}_w(E) = \int_E dF_w(x) \) on measurable \( E \subset [0, 1] \). The inverse of \( F_w(x) \) is also a continuous everywhere increasing function which we call \( G_w(y) \) with corresponding measure \( \mathcal{G}_w(E) \). Let

\[
I(n, j) = [x(n, j - 1), x(n, j)],
\]
\[
J(n, j) = [y(n, j - 1), y(n, j)]
\]

and

\[
I(n, x) = I(n, j), J(n, x) = J(n, j) \text{ for that } j \text{ for which } x \in I(n, j).
\]

\( I(n, y) \) and \( J(n, y) \) are defined similarly. Let \( I^*(n, 2j + \epsilon) = [0, x^*(n, 2j + 1)] \) or \( [x^*(n, 2j + 1), 1] \) and \( J^*(n, 2j + \epsilon) = [0, y^*(n, 2j + 1)] \) or \( [y^*(n, 2j + 1), 1] \) according as \( \epsilon \) equals 0 or 1. We shall write \(|I|\) for the length of the interval \( I \), and \( h(a, b) \) for the function on \( S \) given by \( h(a, b) = a \log b + (1 - a) \log_2 (1 - b) \). All logarithms are taken to the base 2. For any function \( k(x, y) \) on \( S \) we set

\[
E_\mu(k(x, y)) = \int_0^1 \int_0^1 k(x, y) d\mu(x, y)
\]

and

\[
\sigma_\mu^2(k(x, y)) = E_\mu([k(x, y) - E_\mu(k(x, y))]^2).
\]

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THEOREM 1. (a) If \( \sigma_n(h(y, x)) < \infty \) then

\[
\lim_{n \to \infty} \frac{\log |I(n, x)|}{n} = E_\mu(h(y, x))
\]

almost everywhere \((F_\omega)\) for almost all \(\omega\).

(b) If \( \sigma_n(h(y, y)) < \infty \) then

\[
\lim_{n \to \infty} \frac{\log |J(n, x)|}{n} = E_\mu(h(y, y))
\]

almost everywhere \((F_\omega)\) for almost all \(\omega\).

(c) If \( \sigma_n(h(x, x)) < \infty \) then

\[
\lim_{n \to \infty} \frac{\log |I(n, y)|}{n} = E_\mu(h(x, x))
\]

almost everywhere \((G_\omega)\) for almost all \(\omega\).

(d) If \( \sigma_n(h(x, y)) < \infty \) then

\[
\lim_{n \to \infty} \frac{\log |J(n, y)|}{n} = E_\mu(h(x, y))
\]

almost everywhere \((G_\omega)\) for almost every \(\omega\).

In the proof we will need the following law of large numbers for martingales.

**Lemma.** If \( f_n \) is \( F_n \)-measurable, where \( F_n \) is an increasing sequence of \( \sigma \)-fields, \( E(|f_n|) < \infty \), \( E(|f_n|^2) = \sigma_n^2 \) with \( \sum_{n=1}^{\infty} \sigma_n^2/n^2 < \infty \), and if \( E(f_n|F_{n-1}) = 0 \) for all \( n \) then \( \lim_{n \to \infty} n^{-1} \sum_{j=1}^{n} f_j = 0 \) almost everywhere.

**Proof.** \( S_n = \sum_{j=1}^{n} f_j/j \) is a martingale, convergent to some limit \( Z \) since \( E(S_n^2) \leq \sum_{j=1}^{n} \sigma_j^2/j^2 \) for all \( n \). Hence

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} f_j = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} j (S_j - S_{j-1}) = \lim_{n \to \infty} \left( S_n - \frac{1}{n} \sum_{j=1}^{n-1} S_j \right)
\]

\[
= Z - Z = 0
\]

with probability one.

**Proof of Theorem 1.** The proofs of all sections of the theorem are the same so we confine ourselves to the first. Since

\[
\frac{1}{n} \log |I(n, x)| = \frac{1}{n} \sum_{k=1}^{n} \log |I^*(k, x)|
\]

the result will follow from the preceding lemma if we can show that \( f_k = \log |I^*(k, x)| - E_\mu(h(y, x)) \) satisfies \( E_\mu(f_k|F_{k-1}) = 0 \) and \( E_\mu(f_k^2) \)
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\[ F_k = \text{field generated by } k(x, \omega) \]

\[ Q = \text{measure on } [0, 1] \times \Omega \text{ defined by } \int k(x, \omega) dQ = E_\omega \left( \int k(x, \omega) dF_\omega(x) \right). \]

Any \( F_{k-1} \) measurable function has the form

\[ g(x, \omega) = \sum_{j=1}^{k-1} g_j x(x, \omega) \]

where \( x_j(x, \omega) \) is 1 or 0 depending on whether \( x \) is in \( I(k-1, j) \) or not so

\[ E_Q(g_k) = E_\omega \left( \sum_{j=1}^{k-1} g_j \int_{I(k-1, j)} (\log | I(k, u) | - E_\mu(h(y, x))) dF_\omega(u) \right) = 0 \]

which shows that \( E_Q(f_k | F_{k-1}) = 0 \). The verification that \( E_Q(f_k) = \sigma^2(h(y, x)) \) is straightforward.

Let \( C_\mu = \{ I_j \} \) be a set of intervals covering \( E \) with \( \max_j | I_j | \leq \mu \).

The \( \alpha \)-dimensional measure of \( E \) is

\[ \Gamma_\alpha(E) = \lim_{\mu \to 0} \operatorname{g.l.b.} \sum_{I_j \in C_\mu} | I_j |^{\alpha}. \]

The Hausdorff-Besicovitch dimension of \( E \) is

\[ \operatorname{dim} E = \inf(\beta \mid \Gamma_\beta(E) = 0) = \sup(\beta \mid \Gamma_\beta(E) = \infty). \]

**Theorem 2.** Under the hypotheses of Theorem 1, for almost all \( \omega \), there exist sets \( K_\omega, L_\omega \) with \( F_\omega(K_\omega) = G_\omega(L_\omega) = 1 \), such that for any sets \( A \) and \( B \) with \( F_\omega(A) > 0 \) and \( G_\omega(B) > 0 \) we have

\[ \dim(K_\omega \cap A) = E_\mu \{ h(y, y) \} / E_\mu \{ h(y, x) \} \]

and

\[ \dim(L_\omega \cap B) = E_\mu \{ h(x, x) \} / E_\mu \{ h(x, y) \}. \]

**Proof.** The proofs of the two statements are identical so we will prove only the first. Call the right-hand side of the first equation \( \alpha \).

We choose an \( \omega \) in none of the exceptional sets of the first theorem. Then from the first two conclusions of the first theorem, there is a set \( K_\omega \) with \( F_\omega(K_\omega) = 1 \), such that \( | J(n, x) | = | I(n, x) |^{\alpha + \epsilon(1)} \)

for all \( x \in K_\omega \). For each \( x \) in \( K_\omega \) we choose that \( I(n, x) \) with smallest \( n \) such that \( | J(n, x) | < \mu \) and \( | J(n, x) | > | I(n, x) |^{\alpha + \epsilon} \) for \( x_1, x_2 \in I(n, x) \) the choice occurs at the same time so the \( I(n, x) \) are disjoint and countable and cover \( K_\omega \). Hence

\[ 1 = \int_{\bigcup I(n, x)} dF_\omega(x) = \sum_{I(n, x)} | J(n, x) | \geq \sum \sum | I(n, x) |^{\alpha + \epsilon} \]

so \( \Gamma_{\alpha + \epsilon}(K_\omega) \leq 1 \), for every \( \epsilon > 0 \), and hence \( \dim K_\omega \leq \alpha \). Let

\[ C(\epsilon_1, \epsilon_2) = \{ x \mid | J(n, x) | > | I(n, x) |^{\alpha - \epsilon} \text{ or } | I(n, x) | < 2^n [E_\mu(h(y, z)) - n] \} \]
for infinitely many $n$. Let $C_n(e_1, e_2)$ be the union of the intervals $I^*(n, x)$ covering $C(e_1, e_2)$ where for $x \in C(e_1, e_2)$ $n$ is the smallest $n$ for which the conditions of $C(e_1, e_2)$ are satisfied with $|I^*(n, x)| \leq \mu$. Since $\cap_{n \to 0} C_n(e_1, e_2) = C(e_1, e_2)$, $\lim_{\mu \to 0} \bar{F}_\mu(C_n(e_1, e_2)) = 0$. Suppose $\bar{F}_\mu(A) = 2s$. Take $\mu$ so small that $\bar{F}_\mu(C_\mu(e_1, e_2)) < s$, let $A^* = A \cap c(C_\mu(e_1, e_2)) \cap K_\mu$ where $c$ indicates complementation, and set $M(x) = \bar{F}(A^* \cap [0, x])$. $M(x)$ is a monotone, continuous function, $M(1) > s$, and $M(x + h) - M(x - h) < (2h)^{\alpha - \epsilon_3}$, where $\epsilon_3$ depends on the choice of $e_1$ and $e_2$. This happens since $M(x)$ increases only on $I(n, j)$ which fail to lie in $C_\mu(e_1, e_2)$. Hence, if $C_\mu = (I_n)$ is a covering of $A^*$ with $|I_n| < \mu$ then

$$s \leq \int_{A^*} dM(x) = \sum_{n} \int_{I_n} dM(x) \leq \sum |I_n|^{\alpha - \epsilon_3}.$$  

Hence $\Gamma_{\alpha - \epsilon_3}(A^*) > s$. By adjusting $e_1$ and $e_2$, we can choose any $\epsilon_3 > 0$ so $\dim A^* \geq \alpha$. Hence, with the previous inequality we have

$$\alpha \leq \dim A^* \leq \dim A \cap K_\mu \leq \dim K_\mu \leq \alpha.$$  

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