


**DISTRIBUTION MODULO 1 AND SETS OF UNIQUENESS**

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A linear set \( E \subset (0, 1) \) is said to be a set of uniqueness (set \( U \)) for trigonometric expansion if no trigonometric series exists (except vanishing identically) which converges to zero in the set \( CE \) complementary to \( E \). Following Nina Bary we shall say that \( E \) is a set of uniqueness “in the wide sense” (set \( U^* \)) if no Fourier-Stieltjes series exists (except vanishing identically) which converges to zero in \( CE \). If \( E \) is a closed set \( U^* \) it means (see [1, Vol. 1, pp. 344–359, Vol. 2, p. 160]) that \( E \) does not carry any measure whose Fourier-Stieltjes coefficients tend to zero. If \( E \) is a closed set \( U \) (i.e. of uniqueness “strict sense”) it means that \( E \) does not carry any measure or pseudo-measure (cf. [2]) with coefficients tending to zero.

**DEFINITION.** A real sequence of numbers \( \{ u_i \}_i \) will be said to be “badly distributed” modulo 1 if there exists at least one characteristic function \( X(x) \) of open interval \( \Delta \subset (0, 1) \) periodic with period 1 such that

\[
\limsup_{k \to \infty} \frac{X(u_1) + \cdots + X(u_k)}{k} < \int_0^1 X(x)\,dx = |\Delta|
\]

when \(|\Delta|\) stands for the length of \( \Delta \).

**REMARK.** It is easy to see that under this hypothesis there exists a \( \Delta \) with rational end-points having the same property.

**THEOREM.** Let \( E \subset (0, 1) \) be a linear set such that there exists an infinite sequence of positive integers \( \{ n_i \}_i \) increasing to infinity, with the
property that for every \( x \in E \), the sequence \( \{ n_k x \} \) is badly distributed modulo 1. Then \( E \) is a set of the type \( U^* \).

We shall make use of the three following known lemmas:

**Lemma I** (see [1, Vol. 2, pp. 145, 160]). In order to prove that a closed set does not carry a nonvanishing measure with Fourier-Stieltjes coefficients tending to zero, it is sufficient to prove that it does not carry a positive measure having this property.

**Lemma II** (see [1, Vol. 2, p. 144]). Let

\[
d\mu \sim \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}
\]

be a Fourier-Stieltjes series and let

\[
X(x) \sim \sum_{n=-\infty}^{\infty} \gamma_n e^{2\pi i n x}
\]

be the Fourier series of the characteristic function \( X(x) \) of an interval \( \Delta \subset (0, 1) \). Then if the Fourier-Stieltjes coefficients \( c_n \) tend to zero as \( n \to \infty \), one has

\[
\lim_{m \to \infty} \int_0^1 X(mx) d\mu = c_0 \gamma_0 = \int_0^1 X(x) dx \cdot \int_0^1 d\mu(x).
\]

**Lemma III** (see [1, Vol. 2, p. 160]). A set \( E \) which is the union of a denumerable infinity of closed sets \( F_n \) each of which is of the type \( U^* \), is also of the type \( U^* \).

**Proof of the Theorem.** Taking into account the remark following the definition of "bad distribution," we see that to every \( x \in E \) corresponds a characteristic function of open interval with rational endpoints, thus belonging to a denumerable family \( \{ X_m(x) \}_{1}^{\infty} \). Let \( E_m \) be the subset of points \( x \) of \( E \) corresponding to the same function \( X_m(x) \). \( E \) is the union of all the sets \( E_m \).

The set \( E_m \) is itself the union of sets \( E_{m,h} \) (\( h \) a positive integer \( 1 \leq h < \infty \)) such that

\[
\frac{X_m(n_1 x) + \cdots + X_m(n_{\kappa} x)}{\kappa} < \int_0^1 X_m(x) dx \quad \text{for} \quad \kappa \geq h.
\]

The set \( E_{m,h} \) is in turn the union of sets \( E_{m,h,s} \), where

\[
\frac{X_m(n_1 x) + \cdots + X_m(n_{\kappa} x)}{\kappa} \leq \int_0^1 X_m(x) dx - \frac{1}{s} \quad (\kappa \geq h)
\]

where \( s \) takes all positive integral values.
Since $X_m(x)$ is lower-semicontinuous, the set $E_{m,h,s}$ is closed. We shall show that it is of the type $U^*$. Suppose, in fact, that it carries a positive (see Lemma I) nonvanishing measure $d\mu$ with Fourier-Stieltjes coefficients tending to zero. Multiplying (1) by $d\mu$ and integrating with respect to $d\mu$ we set

\[
\int X_m(n_1x)\,d\mu + \cdots + \int X_m(n_\kappa x)\,d\mu
\]

\[
\kappa
\]

\[
\leq \left( \int_0^1 X_m dx \right) \left( \int_0^1 d\mu(x) \right) - \frac{\int_0^1 d\mu}{s} \quad (\kappa \geq h).
\]

Since for $\kappa \to \infty$ the first member tends (Lemma II) to $\int U_m dx \cdot \int_0^1 d\mu(x)$, this leads to a contradiction, and $E_{m,h,s}$ is a $U^*$ set.

It is now enough to use Lemma III to prove the theorem since $E$ is the union of the denumerable family $E_{m,h,s}$ ($m, h, s$ positive integers).

**APPLICATION.** Consider the set of numbers in $(0, 1)$ written in the dyadic system $x = \varepsilon_1/2 + \cdots + \varepsilon_K/2^K + \cdots$ ($\varepsilon_k = 0, 1$) having the property that

\[
\limsup_{K \to \infty} \frac{\varepsilon_1 + \cdots + \varepsilon_K}{K} < \frac{1}{2}.
\]

This set is of the type $U^*$. This is an immediate consequence of our theorem if we remark that $\varepsilon_k = X(2^k x)$ when $X(x)$ is the characteristic function of the interval $(1/2, 1)$. (Here the family $X_m$ is reduced to a single interval.)

**BIBLIOGRAPHY**


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*Added in proof.* The application given here is nothing but the easy part of an important paper of I. I. Pyatetskii-Shapiro (Moskov. Gos. Univ. Uč. Zap. 165, Mathematika 7 (1954), 79–97), where he constructs a set $U^*$ which is not a set $U$. It suggests the following question, that the authors were not able to solve: is the set of non-normal numbers $x$ (i.e., numbers $x$ such that $\limsup (\varepsilon_1 + \cdots + \varepsilon_K)/K > 1/2$ or $\liminf <1/2$) a set $U^*$? In other words, is it a set of measure zero with respect to every positive measure whose Fourier coefficients tend to zero at infinity?