ON HEIGHTS IN NUMBER FIELDS¹

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Let \( K \) be a number field, of degree \( N \) over \( \mathbb{Q} \).

Let \( S_\alpha \) be the set of archimedean absolute values of \( K \), normalized to extend the ordinary absolute value on \( \mathbb{Q} \). For \( x \in K^* \), \( v \in S_\alpha \), put

\[
\|x\|_v = |x|^N_v,
\]

where \( N_v \) is the local degree \([K_v: \mathbb{Q}_v]\), so \( N_v = 1 \) or 2.

For \( X = (X_1, \ldots, X_m) \in K^m \), put

\[
\|X\|_v = \sup \|X_i\|_v, \quad H_\infty(X) = \prod_{v \in S_\alpha} \|X\|_v; \quad \text{and let } [X] \text{ denote the fractional ideal generated by } X_1, \ldots, X_m. \text{ Then the height of } X \text{ is } N[X]^{-1}H_\infty(X).
\]

The class of a point in projective space \( P^{m-1}(K) \) is the class (modulo principal ideals) of the fractional ideal generated by homogeneous coordinates for \( X \).

**Theorem 1.** The number of points in \( P^{m-1}(K) \) of a given class, with height at most \( B \), is

\[
\frac{\kappa_m}{\xi_K(m)} B^m + O(B^{m-1/N}),
\]

where

\[
\kappa_m = \left( \frac{2n(2\pi)^{m^2}}{\sqrt{d}} \right)^m \frac{R}{\omega^m} m^n;
\]

except that for \( m = 2, N = 1 \), the error term is to be replaced by \( O(B \log B) \).

The notation is standard (cf. [1]). For a discussion of the setting of the problem, see [2, Chapter III]. The burden of the proof is carried by Theorem 2.

An \( S_\alpha \)-divisor, or simply divisor on \( K \) is a pair \( b = (a, B) \), where \( a \) is a nonzero fractional ideal and \( B \) is a positive real number. The norm of \( b \) is \( \|b\| = Na^{-1}B \). Map \( K^m - 0^m \) to the group of divisors by \( b_x = ([X], H_\infty(X)) \). For \( m = 1 \), this is a homomorphism, with kernel \( U \) (the group of units of \( K \)) and image the principal divisors.

Let \( U \) act on \( K^m \) by componentwise multiplication. Then associated to any divisor \( b \) is an \( S_\alpha \)-parallelotope \( L_m(b) \subseteq (K^m - 0^m)/U \); it is the set of all orbits \( UX \) for which \( b_x \leq b \), that is: \([X] \subseteq a, H_\infty(X) \leq B \). Similarly, the restricted \( S_\alpha \)-parallelotope \( L'_m(b) \) is the set of all \( UX \) for which \([X] = a, H_\infty(X) \leq B \). Let \( \lambda_m, \lambda'_m \) be the cardinalities of \( L_m, L'_m \).

¹ Details and related results will appear in a forthcoming paper.
Remark 1. \( \lambda_m(b), \lambda'_m(b), \|b\| \) depend only on the class of \( b \) modulo principal divisors.

Remark 2. \( \lambda_m(b) = \lambda'_m(b) = 0 \) for \( \|b\| < 1 \).

Theorem 2. \( \lambda_m(b) = \kappa_m\|b\|^m + O(\|b\|^{m-1/N}) \).

One may restrict \( b \) to range over divisors \( (a_i, B) \), where \( a_i \) are representatives for the ideal classes, by Remark 1; thus it suffices to consider divisors with \( a \) fixed. For \( m = 1 \), \( L_1(a, B) \) is the set of principal ideals contained in \( a \), of norm at most \( B \). Hence Theorem 2 reduces in this case to a classical theorem due to Dedekind and Weber [3].

The reduction of Theorem 1 to Theorem 2 is based on two easy observations. First, there is a bijection from \( L'_m(a, BN\alpha) \) to the set of points of \( \mathbb{P}^{m-1}(K) \) of class \( Cl(a) \), by \( UX \rightarrow K^*X \), so that the problem is to estimate \( \lambda'_m \). Second, \( L_m(a, B) = \bigcup L'_m(ab, B) \), the (disjoint) union extending over all integral ideals \( b \). Thus

\[
\lambda_m(a, B) = \sum \lambda'_m(ab, B).
\]

(The sum is finite, since \( \lambda'_m(ab, B) = 0 \) for \( N\bar{b} > BN\alpha^{-1} \), by Remark 2.) A variant of the Möbius inversion formula gives

\[
\lambda'_m(a, B) = \sum \mu(b)\lambda_m(ab, B).
\]

By Theorem 2, the sum on the right is

\[
\sum \mu(b) \left( \kappa_m \left( \frac{\|b\|}{N\bar{b}} \right)^m + O \left( \left( \frac{\|b\|}{N\bar{b}} \right)^{m-1/N} \right) \right),
\]

summed over all integral \( b \) with \( N\bar{b} \leq \|b\| = BN\alpha^{-1} \). The first term contributes

\[
\kappa_m \|b\|^m \left( \frac{1}{\zeta_K(m)} + O(\|b\|^{1-m}) \right),
\]

and the second is easily estimated to yield Theorem 1.

Bibliography