AN OBSTRUCTION TO FINITENESS OF CW-COMPLEXES

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A cell structure is a convenient means of describing a space; thus it is important to reduce such a structure to a simpler one when possible. For example, it remains unsolved whether a compact topological manifold (or more generally, ANR) has the homotopy type of a finite CW-complex. According to Milnor [2], this would follow from the conjecture that any CW-complex which is dominated by a finite complex has the homotopy type of a finite complex, but we show below that this is false.

Let \( X \) be a connected CW-complex, with universal cover \( \tilde{X} \), and fundamental group \( \pi \) with (integral) group ring \( \Lambda \). Consider the following conditions:

(i) \( X \) is dominated by a complex of finite type (i.e., one with a finite number of cells of each dimension),

(ii) \( \pi \) and all \( H_i(\tilde{X}) \) are countable,

(iii) For \( N < i \), \( H_i(\tilde{X}) = 0 \) and \( H^i(X; \mathcal{B}) = 0 \) for all coefficient bundles \( \mathcal{B} \) (in the sense of Steenrod; generalised to non-abelian coefficients if \( i = 2 \)).

Our results are as follows:

(A) If (i) holds, \( X \) is homotopy equivalent to a complex of finite type.

(B) If \( \Lambda \) is noetherian, (i) is equivalent to: \( \pi \) is finitely presented, and all \( H_i(\tilde{X}) \) are finitely generated \( \Lambda \)-modules.

(C) If \( X \) is dominated by a countable complex, it is homotopy equivalent to one; this condition is equivalent to (ii).

(E) If (iii) holds, and \( N \neq 2 \), \( X \) has the homotopy type of an \( N \)-dimensional complex, countable if (ii) holds.

(F) \( X \) is dominated by a finite complex if and only if (i) and some (iii)\( N \) hold. When this is the case, and \( N \geq 2 \), there is an obstruction \( \theta(X) \) in the projective class group \( \mathcal{K}^0(\Lambda) \), which depends only on the homotopy type of \( X \), and is zero for \( X \) finite. If \( \theta(X) = 0 \), \( X \) has the homotopy type of a finite complex of dimension \( N \).

The proofs are mostly by induction; we obtain complexes \( K^r \) and \( r \)-connected maps \( \phi: K \rightarrow X \), where \( K \) is finite in (A), countable in (C). We then prove that \( \pi_{r+1}(\phi) \) is finitely generated (over \( \Lambda \)) in (A),
and countable in (C), and that we can always use a set of \( \Lambda \)-generators \( (r \geq 2) \) of \( \pi_{r+1}(\phi) \) to attach \((r+1)\)-cells to \( K \), and extend \( \phi \) over them, to obtain an \((r+1)\)-connected map. If \( X \) satisfies (iii)\(_N\), and \( r = N - 1 \), then \( \pi_N(\phi) \) is a projective \( \Lambda \)-module; when it is free, the process above gives a homotopy equivalence.

The crucial step in the proof of (A), which is used again in (F) in showing that \( \theta(X) \) is well defined, is the following lemma of Whitehead [5]:

Let \( P \) be a finite connected complex, \( K \) a connected subcomplex with \( \pi_r(P, K) = 0 \) for \( 1 \leq r < n \). Then there is a formal deformation (and so homotopy equivalence) \( D: P \rightarrow Q \) rel \( K \) such that for \( r < n \), \( Q \) has no \( r \)-cells outside \( K \), and for \( r \geq n + 2 \), \( Q \) has the same number of \( r \)-cells outside \( K \) as \( P \) does.

We observe that there is an interesting analogy between our obstruction in \( K^0(\Lambda) \) (which is the Grothendieck group of finitely generated projective modulo free modules) to existence of finite complexes equivalent to \( X \), and Whitehead's obstruction in \( K^1(\Lambda) \) (reduced by \( \pm \pi \)) to their uniqueness up to formal deformation [5]. We refer the reader to Bass and Schanuel [1] for the relation between \( K^0(\Lambda) \) and \( K^1(\Lambda) \).

According to Swan [4], \( K^0(\Lambda) \) is finite, if \( \pi \) is, and by Rim [3], if \( \pi \) is cyclic of prime order, \( K^0(\Lambda) \) is isomorphic to the ideal class group of the corresponding cyclotomic field. This gives several examples both of zero and of nonzero \( K^0(\Lambda) \).

The main unsatisfactory feature of the above is our inability to construct 2-dimensional complexes under appropriate hypotheses. Roughly speaking, by the time we have enough 2-cells to give relations between the generators of the fundamental group, we may have too many for the homology.

References


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