A PROBLEM IN PARTITIONS RELATED TO THE STIRLING NUMBERS

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Let

$$S(n, r) = \frac{1}{r!} \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} s^n$$

denote the Stirling number of the second kind and put

$$A_n(x) = \sum_{r=0}^{n} S(n, r) x^r.$$

In a recent paper [1] the writer has determined the factorization (mod 2) of the polynomial $A_n(x)$.

Put

$$c_{nr} = S(n + 1, r + 1);$$

then we have

$$c_{n,2r} \equiv \binom{n-r}{r} \pmod 2 \qquad (0 \leqq 2r < n),$$

$$c_{n,2r+1} \equiv \binom{n-r-1}{r} \pmod 2 \qquad (2r + 1 \leqq n).$$

For fixed n, let $\theta_0(n)$ denote the number of odd $c_{n,2r}$ and $\theta_1(n)$ the number of even $c_{n,2r}$. Then

$$\theta_0(2n + 1) = \theta_0(n), \qquad \theta_0(2n) = \theta_0(n) + \theta_0(n - 1)$$

and

$$\theta_1(n + 1) = \theta_0(n).$$

Moreover we have the generating function

$$\sum_{n=0}^{\infty} \theta_0(n) x^n = \prod_{n=0}^{\infty} (1 + x^{2^n} + x^{2^{n+1}}).$$

It follows that $\theta_0(n)$ can also be defined as the number of partitions

$$n = n_0 + n_1 \cdot 2 + n_2 \cdot 2^2 + \cdots \qquad (0 \leqq n_j \leqq 2)$$

subject to the conditions

(i) if $n_0=1$ then $n_1 \leqq 1$,
(ii) if $n_1=2$ then $n_2 \leqq 1$,
(iii) if $n_2=2$ then $n_3 \leqq 1$,
and so on.

The first few values of $\theta_0(n)$ are given in the following table.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\theta_0(n)$	1	1	2	1	3	2	3	1	4	3	5	2	5	3	5	1

In the present paper the following additional properties of $\theta_0(n)$ are obtained.

$$\theta_0(2^r m) = \theta_0(m) + r\theta_0(m-1) \qquad (r \geqq 0,\ m \geqq 1), \tag{1}$$

in particular

$$\theta_0(2^r) = r + 1; \tag{2}$$

$$\theta_0(2^r m - 1) = \theta_0(m-1) \qquad (r \geqq 0,\ m \geqq 1), \tag{3}$$

in particular

$$\theta_0(2^r - 1) = 1. \tag{4}$$

$$\theta_0(2^{r+s} + 2^r) = rs + r + s \qquad (r \geqq 0,\ s \geqq 1), \tag{5}$$

$$\theta_0(2^{r+s} - 2^r) = rs + 1 \qquad (r \geqq 0,\ s \geqq 1), \tag{6}$$

$$\theta_0(2^{r+s+t} + 2^{r+s} + 2^r) = (r+1)s + rt + (r+1)st - 1 \qquad (r \geqq 0,\ s \geqq 1,\ t \geqq 1). \tag{7}$$

Generally if

$$n = 2^{r_0} + 2^{r_0+r_1} + \cdots + 2^{r_0+r_1+\cdots+r_k}, \tag{8}$$

where $r \geqq 0$, $r \geqq 1, \cdots, r_k \geqq 1$ and

$$n_j = 2^{r_j} + 2^{r_j+r_{j+1}} + \cdots + 2^{r_j+\cdots+r_k} \qquad (0 \leqq j \leqq k), \tag{9}$$

so that $n_0 = n$, then we have

$$\theta_0(n) = (1 + r_0)\theta_0(n_1) - \theta_0(n_2). \tag{10}$$

We may think of (10) as a recursion formula. With the notation (9) we have

$$\theta_0(n_j) = (1 + r_j)\theta_0(n_{j+1}) - \theta_0(n_{j+2}) \qquad (0 \leqq j < k), \tag{11}$$

where $n_{k+1} = 0$. If we put

(12) $$m_j = 2^{r_0} + 2^{r_0+r_1} + \cdots + 2^{r_0+r_1+\cdots+r_j},$$

then we have the companion formula

(13) $$\theta_0(m_j) = (1 + r_j)\theta_0(m_{j-1}) - \theta_0(m_{j-2}) \qquad (j \geqq 1),$$

where $m_{-1}=0$. Indeed (10) and (13) are equivalent. A more general relation is

(14) $$\theta_0(n) = \theta_0(m_j)\theta_0(n_{j+1}) - \theta_0(m_{j-1})\theta_0(n_{j+2}) \qquad (0 \leqq j \leqq k),$$

where $m_{-1}=n_{k+1}=0$.

The recurrence (13) suggests a connection with continuants [2, pp. 466–474]. Let

$$K\begin{pmatrix} & b_1, \cdots, b_k \\ a_0, & a_1, \cdots, a_k \end{pmatrix}$$

denote a continuant. Then we have

(15) $$\theta_0(n) = K\begin{pmatrix} & -1, \cdots, -1 \\ 1 + r_0, & 1 + r_1, \cdots, 1 + r_k \end{pmatrix}.$$

This may be written briefly in the form

$$\theta_0(n) = K'(1 + r_0, 1 + r_1, \cdots, 1 + r_k).$$

From known properties of continuants we have for example

$$K'(1 + r_0, 1 + r_1, \cdots, 1 + r_k) = K'(1 + r_k, 1 + r_{k-1}, \cdots, 1 + r_k).$$

Also we are led to a determinantal representation of $\theta_0(n)$.

When $r_0=r_1=\cdots=r_k=r$ we get the following explicit result:

(16) $$\theta_0(n) = \frac{\epsilon^{k+2} - \epsilon^{-k-2}}{\epsilon - \epsilon^{-1}} \qquad (r \neq 1),$$

where

$$n = 2^r(2^{(k+1)r} - 1)/(2^r - 1)$$

and

(17) $$\epsilon = \tfrac{1}{2}\{1 + r + \sqrt{(1 + r)^2 - 4}\}.$$

For $r=1$, however, we get $\theta_0(n)=k+2$, which is equivalent to

$$\theta_0(2(2^k - 1)) = k + 1.$$

Finally we consider some questions of a different kind. The equation

$$\theta_0(n) = 1 \tag{18}$$

holds if and only if $n=2^k-1$. The equation

$$\theta_0(n) = 2 \tag{19}$$

holds if and only if

$$n = 2^r + 2^{r-1} - 1 \qquad (r = 1, 2, 3, \cdots).$$

The equation

$$\theta_0(n) = 3 \tag{20}$$

is satisfied if and only if n is of one of the forms

$$2^{r+1} + 2^{r-1} - 1 \qquad (r = 1, 2, 3, \cdots),$$
$$3 \cdot 2^r + 2^{r-1} - 1 \qquad (r = 1, 2, 3, \cdots).$$

The equation

$$\theta_0(n) = t \qquad (t \geqq 3), \tag{21}$$

where t is assigned, is always solvable. There exist a finite number of *even* solutions $e_1, e_2, \cdots, e_w$ such that all solutions are given by

$$e_j,\ 2e_j + 1,\ 4e_j + 3,\ 8e_j + 7, \cdots \qquad (1 \leqq j \leqq w). \tag{22}$$

An upper bound for $\theta_0(n)$ is given by

$$\theta_0(n) \leqq (1 + r_0)(1 + r_1) \cdots (1 + r_k); \tag{23}$$

another bound is

$$\theta_0(n) < \left(\frac{\log_2 n}{k + 1}\right)^{k+1}. \tag{24}$$

When $r_0=r_1= \cdots =r_k=r$, the bound (24) cannot be improved; indeed

$$\theta_0(n) \sim r^{k+1} \qquad (r \to \infty). \tag{25}$$

A fuller account of these results will appear elsewhere.

References

1. L. Carlitz, *Single variable Bell polynomials*, Collect. Math. **14** (1962), 13–25.
2. G. Chrystal, *Algebra*. II, Edinburgh, 1889.

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