A NOTE ON THE ADJOINT OF A PERTURBED OPERATOR

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Let $X$ and $Y$ be Banach spaces. By an operator from $X$ to $Y$ we mean a linear operator with domain $D(T) \subseteq X$ and range $R(T) \subseteq Y$. An operator $T$ from $X$ to $Y$ is said to be a Fredholm operator if $T$ is closed, the null space $N(T)$ has finite dimension, and the range $R(T)$ is closed and has finite codimension in $Y$. We denote by $\Phi(X, Y)$ the set of all Fredholm operators from $X$ to $Y$. If $T \in \Phi(X, Y)$, the index of $T$ is defined to be

$$\text{ind}(T) = \dim(N(T)) - \text{codim}(R(T)).$$

Suppose $T \in \Phi(X, Y)$. Since $T$ is closed, the graph $G(T)$ is a closed subspace of $X \times Y$. An operator $C$ from $X$ to $Y$ is said to be $T$-compact if $C$ is closable, $D(C) \supseteq D(T)$, and the mapping $(x, Tx) \mapsto Cx$ is compact as an operator from $G(T)$ into $Y$.

The following results are well known:

1. If $T \in \Phi(X, Y)$ and $C$ is $T$-compact, then $T + C \in \Phi(X, Y)$ and $\text{ind}(T + C) = \text{ind}(T)$. (See [2, Theorem 2.6].)

2. If $T$ is a closed operator from $X$ to $Y$ and $D(T)$ is dense in $X$, then $T$ is in $\Phi(X, Y)$ if and only if the adjoint operator $T^*$ is in $\Phi(X^*, Y^*)$. If so, then $\text{ind}(T^*) = -\text{ind}(T)$. (For by [1, Theorem A], $R(T)$ is closed if and only if $R(T^*)$ is closed. If so, then $\dim(N(T^*)) = \text{codim}(R(T))$ and $\dim(N(T)) = \text{codim}(R(T^*))$.)

**Lemma.** Suppose $S \in \Phi(X, Y)$, $T \in \Phi(X, Y)$, and $S \subseteq T$. Then $\text{ind}(S) \leq \text{ind}(T)$, and equality holds if and only if $S = T$.

**Proof.** If $S \subseteq T$, then $N(S) \subseteq N(T)$ and $R(S) \subseteq R(T)$, so the inequality is obvious. Equality implies $N(S) = N(T)$ and $R(S) = R(T)$, so $S = T$.

**Proposition.** Suppose $T \in \Phi(X, Y)$ and $D(T)$ is dense in $X$. If $C$ is an operator from $X$ to $Y$ such that $C$ is $T$-compact and $C^*$ is $T^*$-compact, then $T^* + C^* = (T + C)^*$.

**Proof.** From (1) and (2) it follows that $T^* + C^* \in \Phi(X^*, Y^*)$ and $(T + C)^* \in \Phi(X^*, Y^*)$. Furthermore

$$\text{ind}(T^* + C^*) = \text{ind}(T^*) = -\text{ind}(T)$$

$$= -\text{ind}(T + C) = \text{ind}((T + C)^*).$$

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It is obvious that $T^* + C^* \subseteq (T + C)^*$, and equality follows from the lemma.

**Remark.** If $C$ is $T$-compact, $C^*$ need not be $T^*$-compact. In fact it can happen that $D(C^*) \cap D(T^*) = (0)$, and therefore $(T + C)^* \neq T^* + C^*$, even when $T$ is the inverse of a positive definite compact Hermitian operator in a Hilbert space.

**References**


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