INVARIANT DOMAINS FOR KLEINIAN GROUPS\textsuperscript{1}

BY R. ACCOLA

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If the limit set, $\Sigma$, of a properly discontinuous group, $\Gamma$, of fractional linear transformations of the Riemann sphere, $S$, contains more than two points, call $\Gamma$ Kleinian. Otherwise, call $\Gamma$ elementary.

Let $\{\Omega_i\}$ be an enumeration of the components of $\Omega$, the set of discontinuity. If $O$ is a domain in $S$, i.e., $O$ is open and connected, let $\Gamma(O)$ be the subgroup of $\Gamma$ of elements which map $O$ onto itself. If $\Gamma(O) = \Gamma$, call $O$ an invariant domain. If $\Gamma(\Omega_i) = \{\text{id}\}$, call $\Omega_i$ an atom.

**Theorem 1.** If $\Gamma$ possesses three disjoint invariant domains then $\Gamma$ is cyclic.

**Theorem 2.** Suppose $\Gamma$ possesses an invariant component $\Omega_0$. If $0 \neq i \neq j \neq 0$, then $\Gamma(\Omega_i) \cap \Gamma(\Omega_j)$ is a nonloxodromic and nonhyperbolic cyclic group. If $\Omega_0$ is simply connected this latter group is nonelliptic.

**Theorem 3.** If $\Gamma$ is a Kleinian group with two disjoint invariant domains, then there exists a maximal pair of disjoint invariant domains each of which is simply connected. All noninvariant components of $\Omega$ are atoms.

The author is grateful to Leon Greenberg for pointing out how the next theorem follows from the methods used in proving the previous theorems and, essentially, from a deep theorem of Nielsen and Fenchel\textsuperscript{2} on Fuchsian groups.

**Theorem 4.** If $O_1$ and $O_2$ are a maximal pair of disjoint invariant domains for a Kleinian group, $\Gamma$, then $O_1/\Gamma$ and $O_2/\Gamma$ are homeomorphic surfaces.

Examples are given where (a) $\Omega$ and $\Sigma$ are both connected and (b) where $\Gamma$ possesses two invariant components and atoms.

The proofs follow from remarks of which the following are typical.

1. A closed set, invariant under $\Gamma$, contains $\Sigma$.
2. The components of the complement of a closed connected set are simply connected.
3. If $O$ is a simply connected domain invariant under a loxodromic transformation, $T$, then there is a Jordan arc in $O$ invariant under $T$.

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\textsuperscript{2} Added in proof. $O_1$ and $O_2$ are a maximal pair of disjoint invariant domains if whenever $O_i'$ and $O_j'$ are a pair of disjoint invariant domains such that $O_i \subset O_i'$, then $\Omega_i = O_i'$ for $i = 1, 2$. 

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DIFFERENTIABLE NORMS IN BANACH SPACES

BY GUILLERMO RESTREPO

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1. Introduction. In [4, p. 28] S. Lang has asked whether or not a separable Banach space has an admissible norm of class \( C^1 \). In this note we indicate a proof of the following theorem, which characterizes those Banach spaces for which such a norm exists.

**Theorem 1.** A separable Banach space has an admissible norm of class \( C^1 \) if and only if its dual is separable.

It follows from this theorem that not even \( C(I) \) possesses an admissible differentiable norm.

2. Preliminaries. Let \( X \) be a Banach space with norm \( \alpha \); we write

\[ S_\alpha = \{ x \mid \alpha(x) = 1 \} \quad \text{and} \quad B_\alpha = \{ x \mid \alpha(x) \leq 1 \}. \]

A norm in \( X \) is admissible if it induces the same topology as does \( \alpha \). The dual space is written \( X^* \) and the norm dual to \( \alpha \) is denoted by \( \alpha^* \). An \( f \in X^* \) is called a support functional to \( B_\alpha \) at \( x \in S_\alpha \) if \( \alpha^*(f) = f \cdot x \); if \( f \) has norm 1, it is called a normalized support functional and is written \( \nu_x \). A norm is smooth if there is a unique normalized support functional to \( B_\alpha \) at each \( x \in S_\alpha \). The norm \( \alpha \) is differentiable at \( x \neq 0 \) if there is an \( \alpha'(x) \in X^* \) such that

\[
\lim_{y \to x; y 
eq x} \frac{|\alpha(y) - \alpha(x) - \alpha'(x) \cdot (y - x)|}{\alpha(y - x)} = 0
\]

connecting the fixed points of \( T \). (4) If \( O_1 \) and \( O_2 \) are disjoint simply connected domains invariant under a loxodromic \( T \), the corresponding arcs, as in (3), divide \( S \) into two Jordan regions, one or the other of which must contain any domain disjoint from \( O_1 \) and \( O_2 \). (5) If \( O \) is a simply connected domain invariant under an elliptic \( T \), then \( O \) must contain a fixed point of \( T \).


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