DIFFERENTIABLE NORMS IN BANACH SPACES

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1. Introduction. In [4, p. 28] S. Lang has asked whether or not a separable Banach space has an admissible norm of class $C^1$. In this note we indicate a proof of the following theorem, which characterizes those Banach spaces for which such a norm exists.

**Theorem 1.** A separable Banach space has an admissible norm of class $C^1$ if and only if its dual is separable.

It follows from this theorem that not even $C(I)$ possesses an admissible differentiable norm.

2. Preliminaries. Let $X$ be a Banach space with norm $\alpha$; we write $S_\alpha = \{x | \alpha(x) = 1\}$ and $B_\alpha = \{x | \alpha(x) \leq 1\}$. A norm in $X$ is admissible if it induces the same topology as does $\alpha$. The dual space is written $X^*$ and the norm dual to $\alpha$ is denoted by $\alpha^*$. An $f \in X^*$ is called a support functional to $B_\alpha$ at $x \in S_\alpha$ if $\alpha^*(f) = f \cdot x$; if $f$ has norm 1, it is called a normalized support functional and is written $\nu_x$. A norm is smooth if there is a unique normalized support functional at each $x \in S_\alpha$. The norm $\alpha$ is differentiable at $x \neq 0$ if there is an $\alpha'(x) \in X^*$ such that

$$\lim_{y \to x; y \neq x} \frac{|\alpha(y) - \alpha(x) - \alpha'(x) \cdot (y - x)|}{\alpha(y - x)} = 0$$

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and a norm differentiable at each \( x \in X - \{0\} \) is of class \( C^1 \) if the map \( \alpha': X - \{0\} \to X^* \), defined by \( x \mapsto \alpha'(x) \), is continuous. The following two results are well known:

1. Klee [3]. Let \( X \) and \( X^* \) be separable. Then there exists an admissible norm \( \alpha \) in \( X \) such that \( \alpha^* \) is strictly convex, and such that whenever a sequence \( \{f_n\} \) in \( X^* \) converges to \( f \in X^* \) in the \( w^* \)-topology, then \( \alpha^*(f_n) \to \alpha^*(f) \) implies \( \alpha^*(f - f_n) \to 0. \)

2. Bishop-Phelps [1]. In any Banach space \( X \), the set of all the support functionals to \( B_\alpha \) is dense in \( X^* \).

3. Proof of Theorem 1. It is not difficult to see that if the norm \( \alpha \) is differentiate at \( x \in S_\alpha \), then \( \alpha'(x) = \nu_x \) is a normalized support functional to \( B_\alpha \) at \( x \), and is unique. The map \( x \mapsto \nu_x \) of \( S_\alpha \) into \( S_\alpha^* \) is denoted by \( \mu \). We first establish the following general theorem:

**Theorem 2.** (a) If \( \alpha \) is a smooth norm in \( X \), then the map \( \mu \) is continuous when the norm topology is used in \( X \) and the \( w^* \)-topology is used in \( X^* \).

(b) The norm \( \alpha \) is of class \( C^1 \) if and only if the map \( \mu \) is continuous in the norm topologies.

(c) A norm is of class \( C^1 \) if and only if it is differentiable at every point of \( S_\alpha \).

Complete details will be published elsewhere; using this result, we prove Theorem 1 as follows:

Assume \( X^* \) is separable, and let \( \alpha \) be the norm of Klee’s theorem. By a well-known duality, \( \alpha \) is smooth. Theorem 2(a) assures \( \mu \) is continuous with the \( w^* \)-topology in \( X^* \), and then Klee’s theorem shows \( \mu \) is continuous in the norm topology. By Theorem 2(b), \( \alpha \) is therefore of class \( C^1 \).

Assume now that \( \alpha \) is of class \( C^1 \). Extend the continuous map \( \mu \) to a continuous \( \hat{\mu}: X - \{0\} \to X^* \) by setting \( \hat{\mu}(x) = \alpha(x) \cdot \nu_x(x/\alpha(x)). \) The image of \( \hat{\mu} \) evidently contains the set of all the support functionals to \( B_\alpha \), and an application of the Bishop-Phelps theorem shows at once that \( X^* \) is separable whenever \( X \) is separable.

**Bibliography**


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