

connecting the fixed points of  $T$ . (4) If  $O_1$  and  $O_2$  are disjoint simply connected domains invariant under a loxodromic  $T$ , the corresponding arcs, as in (3), divide  $S$  into two Jordan regions, one or the other of which must contain any domain disjoint from  $O_1$  and  $O_2$ . (5) If  $O$  is a simply connected domain invariant under an elliptic  $T$ , then  $O$  must contain a fixed point of  $T$ .

The examples are elaborations of the ideas in L. R. Ford, *Automorphic functions*, 2nd ed., Chelsea, 1951, pp. 55–59.

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## DIFFERENTIABLE NORMS IN BANACH SPACES<sup>1</sup>

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**1. Introduction.** In [4, p. 28] S. Lang has asked whether or not a separable Banach space has an admissible norm of class  $C^1$ . In this note we indicate a proof of the following theorem, which characterizes those Banach spaces for which such a norm exists.

**THEOREM 1.** *A separable Banach space has an admissible norm of class  $C^1$  if and only if its dual is separable.*

It follows from this theorem that not even  $C(I)$  possesses an admissible differentiable norm.

**2. Preliminaries.** Let  $X$  be a Banach space with norm  $\alpha$ ; we write  $S_\alpha = \{x | \alpha(x) = 1\}$  and  $B_\alpha = \{x | \alpha(x) \leq 1\}$ . A norm in  $X$  is admissible if it induces the same topology as does  $\alpha$ . The dual space is written  $X^*$  and the norm dual to  $\alpha$  is denoted by  $\alpha^*$ . An  $f \in X^*$  is called a support functional to  $B_\alpha$  at  $x \in S_\alpha$  if  $\alpha^*(f) = f \cdot x$ ; if  $f$  has norm 1, it is called a normalized support functional and is written  $\nu_x$ . A norm is smooth if there is a unique normalized support functional to  $B_\alpha$  at each  $x \in S_\alpha$ . The norm  $\alpha$  is differentiable at  $x \neq 0$  if there is an  $\alpha'(x) \in X^*$  such that

$$\lim_{y \rightarrow x; y \neq x} \frac{|\alpha(y) - \alpha(x) - \alpha'(x) \cdot (y - x)|}{\alpha(y - x)} = 0$$

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and a norm differentiable at each  $x \in X - \{0\}$  is of class  $C^1$  if the map  $\alpha': X - \{0\} \rightarrow X^*$ , defined by  $x \rightarrow \alpha'(x)$ , is continuous. The following two results are well known:

1. Klee [3]. Let  $X$  and  $X^*$  be separable. Then there exists an admissible norm  $\alpha$  in  $X$  such that  $\alpha^*$  is strictly convex, and such that whenever a sequence  $\{f_n\}$  in  $X^*$  converges to  $f \in X^*$  in the  $w^*$ -topology, then  $\alpha^*(f_n) \rightarrow \alpha^*(f)$  implies  $\alpha^*(f - f_n) \rightarrow 0$ .

2. Bishop-Phelps [1]. In any Banach space  $X$ , the set of all the support functionals to  $B_\alpha$  is dense in  $X^*$ .

3. **Proof of Theorem 1.** It is not difficult to see that if the norm  $\alpha$  is differentiable at  $x \in S_\alpha$ , then  $\alpha'(x) = \nu_x$  is a normalized support functional to  $B_\alpha$  at  $x$ , and is unique. The map  $x \rightarrow \nu_x$  of  $S_\alpha$  into  $S_{\alpha^*}$  is denoted by  $\mu$ . We first establish the following general theorem:

**THEOREM 2.** (a) *If  $\alpha$  is a smooth norm in  $X$ , then the map  $\mu$  is continuous when the norm topology is used in  $X$  and the  $w^*$ -topology is used in  $X^*$ .*

(b) *The norm  $\alpha$  is of class  $C^1$  if and only if the map  $\mu$  is continuous in the norm topologies.*

(c) *A norm is of class  $C^1$  if and only if it is differentiable at every point of  $S_\alpha$ .*

Complete details will be published elsewhere; using this result, we prove Theorem 1 as follows:

Assume  $X^*$  is separable, and let  $\alpha$  be the norm of Klee's theorem. By a well-known duality,  $\alpha$  is smooth. Theorem 2(a) assures  $\mu$  is continuous with the  $w^*$ -topology in  $X^*$ , and then Klee's theorem shows  $\mu$  is continuous in the norm topology. By Theorem 2(b),  $\alpha$  is therefore of class  $C^1$ .

Assume now that  $\alpha$  is of class  $C^1$ . Extend the continuous map  $\mu$  to a continuous  $\hat{\mu}: X - \{0\} \rightarrow X^*$  by setting  $\hat{\mu}(x) = \alpha(x) \cdot \mu(x/\alpha(x))$ . The image of  $\hat{\mu}$  evidently contains the set of all the support functionals to  $B_\alpha$ , and an application of the Bishop-Phelps theorem shows at once that  $X^*$  is separable whenever  $X$  is separable.

#### BIBLIOGRAPHY

1. E. Bishop and R. R. Phelps, *The support functionals of a convex set*, Proc. Sympos. Pure Math., Vol. 7, pp. 27-35, Amer. Math. Soc., Providence, R. I., 1963.
2. J. Dieudonné, *Foundations of modern analysis*, Academic Press, New York, 1960.
3. V. Klee, *Mappings into normed linear spaces*, Fund. Math. 49 (1960), 25-63.
4. S. Lang, *Introduction to differentiable manifolds*, Interscience, New York, 1962.

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