LOCALLY FLAT STRINGS

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I. The Schoenflies Theorem for strings. In [1], Stallings defines a string of type \((n, k)\) to be a pair \((R^n, Y)\), where \(Y\) is a closed subset of \(R^n\) such that \(Y\) is homeomorphic to \(R^k\). Similarly, he defines a pair \((S^n, X)\), where \(X\) is homeomorphic to \(S^k\), to be a knot of type \((n, k)\). A pair \((A, X)\) of \((n, k)\)-manifolds is said to be locally smooth if each point of \(X\) has a neighborhood \(U\) in \(A\) such that the pair \((U, U\cap X)\) is homeomorphic to the pair \((R^n, R^k)\). Thus, his definition of locally smooth is equivalent to Brown's [2] definition of locally flat.

Let \((R^n, F)\) be a locally smooth string of type \((n, n-1)\); \(F\) separates \(R^n\) into two components whose closures are \(A\) and \(B\). In [1], Stallings states that it seems possible that either \(A\) or \(B\) must be homeomorphic to a closed half-space of \(R^n\). Harrold and Moise [3] have proved this for \(n=3\). In this note we observe that both \(A\) and \(B\) are closed half-spaces of \(R^n\) for \(n>3\) and hence we have a Schoenflies theorem for strings of type \((n, n-1)\) for \(n>3\).

**Theorem 1.1.** Let \((R^n, Y)\) be a locally flat string of type \((n, n-1)\) and let \(A\) and \(B\) be the closures of the complementary domains of \(Y\) in \(R^n\). Then \(A\) and \(B\) are homeomorphic to a closed half-space of \(R^n\) for \(n>3\).

**Corollary 1.2.** Let \((R^n, Y)\) be a locally flat string of type \((n, n-1)\) for \(n>3\). Then \((R^n, Y)\) is trivial, that is, there is a homeomorphism \(h\) of \((R^n, Y)\) onto \((R^n, R^{n-1}\times\{0\})\).

**Corollary 1.3.** Let \(f_1, f_2\) be two locally flat embeddings of \(R^{n-1}\) as a closed subset of \(R^n\) for \(n>3\). Then there is a homeomorphism \(h\) of \(R^n\) onto \(R^n\) such that \(hf_1=f_2\).

Theorem 1.1 follows immediately from a recent result of Cantrell's [4]. Cantrell showed that a knot \((S^n, Y)\) of type \((n, n-1)\) is trivial for \(n>3\) provided \(Y\) is locally flat except at one point. Thus, if \((R^n, X)\) is a locally flat string of type \((n, n-1)\) and \((S^n, Y)\) is the one point compactification of \((R^n, X)\), \(Y\) is locally flat except at the compactification point. Hence \((S^n, Y)\) is trivial for \(n>3\) and Theorem 1.1 follows.

II. The Slab Conjecture. In this section we consider the relationship of locally flat strings of type \((n, n-1)\) to the Annulus Conjecture. We now state the Annulus Conjecture.
II.1. The Annulus Conjecture. Let $S_1^{n-1}, S_2^{n-1}$ be two disjoint locally flat $n-1$ spheres embedded in $S^n$. Then the submanifold $M$ of $S^n$ bounded by $S_1^{n-1} \cup S_2^{n-1}$ is homeomorphic to $S^{n-1} \times [0, 1]$.

Although the Annulus Conjecture is unsolved for $n > 3$, the following theorem which is well known but does not seem to be in print holds.

**Theorem II.2.** Let $S_1^{n-1}, S_2^{n-1}$ be two disjoint locally flat $n-1$ spheres embedded in $S^n$. Then if $M$ is the submanifold of $S^n$ bounded by $S_1^{n-1} \cup S_2^{n-1}$ and $M_i = M - S_i^{n-1}$, $M_i$ is homeomorphic to $S^{n-1} \times [0, 1]$ for $i = 1, 2$.

**Proof.** Let $A_i$ be the closed $n$-cell [2] with boundary $S_i^{n-1}$ such that $A_i \cap M = S_i^{n-1}$ for $i = 1, 2$. $A_i$ is cellular and hence by Theorem I of [5], $S^n/A_i$ is homeomorphic to $S^n$ and the theorem follows.

A theorem analogous to Theorem II.2 holds for locally flat strings of type $(n, n-1)$ for $n > 3$.

**Theorem II.3.** Let $R_1^{n-1}, R_2^{n-1}$ be two disjoint locally flat $n-1$ planes embedded as closed subsets of $R^n$ for $n > 3$. Then if $M$ is the submanifold of $R^n$ bounded by $R_1^{n-1} \cup R_2^{n-1}$ and $M_i = M - R_i^{n-1}$, $M_i$ is homeomorphic to $R^{n-1} \times [0, 1]$ for $i = 1, 2$.

**Proof.** In view of Corollary I.2, we may assume that $R_1^{n-1} = R^{n-1} \times 0$ and $R_2^{n-1} \subset R^{n-1} \times (0, \infty)$. Let $A_2$ be the closed half-space (by Theorem I.1) of $R^n$ bounded by $R_2^{n-1}$ which does not contain $R_1^{n-1}$. By Theorem I.1, $R^n - A_2$ is homeomorphic to $R^n$ and hence by the same theorem $M_2$ is homeomorphic to $R^{n-1} \times [0, 1)$. Similarly, $M_1$ is homeomorphic to $R^{n-1} \times [0, 1)$.

We now state the Slab Conjecture.

II.4. The Slab Conjecture. Let $R_1^{n-1}, R_2^{n-1}$ be disjoint locally flat $n-1$ planes embedded as closed subsets of $R^n$. Then if $M$ is the submanifold of $R^n$ bounded by $R_1^{n-1} \cup R_2^{n-1}$, $M$ is homeomorphic to $R^{n-1} \times [0, 1]$.

It should be noted that the Slab Conjecture is false in dimension 3. A counterexample can be obtained as follows. Let $S_1^2$ be the 2-sphere boundary of a 3-cell obtained by "swelling" a Fox-Artin arc (Example 1.2) [6]. We may assume that $S_1^2$ is contained in the unit 3-ball $B^3$ of $S^3$, that $S_1^2 \cap B^3 = \varnothing$, and that $S_1^2$ is locally flat at each point other than $p$. Let $S_2^3 = B^3 \setminus p$, $R_2^2 = S_2^2 - p$ and $R_2^2 = S_2^2 - p$. Then $R_2^3, R_2^2$ are disjoint locally flat 2-planes embedded as closed subsets of $R^3 = S^3 - p$. The 3-dimensional Slab Conjecture would imply that the closure of the complementary domain of $S_1^3$ in $S^3$ containing $R_2^2$ is a closed 3-cell which is a contradiction since $S_1^3$ is wild in $S^3$.
The Slab Conjecture is unsolved for \( n > 3 \) and the following theorem indicates that it is possibly stronger than the Annulus Conjecture.

**Theorem II.5.** The Slab Conjecture implies the Annulus Conjecture for \( n > 3 \).

**Proof.** Let \( S_1^{n-1}, S_2^{n-1} \) be disjoint locally flat \( n-1 \) spheres embedded in \( S^n \). In view of Brown's theorem \([2]\), we may assume that \( S_1^{n-1} = \) the equator of \( S^n \) and \( S_2^{n-1} \) lies in the northern hemisphere of \( S^n \). Now there is a unique \( n-1 \) sphere \( S_\beta^{n-1} \) with the following properties:

1. \( S_\beta^{n-1} \) lies in the northern hemisphere of \( S^n \).
2. \( S_\beta^{n-1} \) is concentric with \( S^{n-1} = \) the equator of \( S^n \).
3. \( S_\beta^{n-1} \cap S_2^{n-1} \) is not empty.
4. The half-open annulus bounded by \( S^{n-1} \cup S_\beta^{n-1} \) but not containing \( S_\beta^{n-1} \) does not intersect \( S_2^{n-1} \).

Let \( p \in S_\beta^{n-1} \cap S_2^{n-1} \) and let \( D^{n-1} \) be the standard unit \( n-1 \) cell in \( S^{n-1} \) with center \( p' \) where \( p' \) and \( p \) lie on a great circle passing through the north pole. Let \( C \) be the cone over the base \( \hat{D}^{n-1} \) with vertex \( p \). Then \( [S_\beta^{n-1} - \text{Int}(D^{n-1})] \cup C = S_3^{n-1} \) is a locally flat \( n-1 \) sphere such that \( S_3^{n-1} \cap S_2^{n-1} = p \).

If we define \( R_1^{n-1} = S_\beta^{n-1} - p \) and \( R_2^{n-1} = S_2^{n-1} - p \), then \( R_1^{n-1}, R_2^{n-1} \) are disjoint locally flat \( n-1 \) planes embedded as closed subsets of \( S^n - p = R^n \). By the Slab Conjecture, the submanifold \( N^n \) bounded by \( R_1^{n-1} \cup R_2^{n-1} \) in \( R^n \) is homeomorphic to \( R^{n-1} \times [0, 1] \). Hence, there is a homeomorphism \( h \) of \( N^n \) onto \( R^{n-1} \times [0, 1] \) where \( h(R_1^{n-1}) = R^{n-1} \times 0 \) and \( h(R_2^{n-1}) = R^{n-1} \times 1 \). Since \( \hat{D}^{n-1} \) is a flat \( n-2 \) sphere in \( R_1^{n-1} \), \( h(\hat{D}^{n-1}) \) is a flat \( n-2 \) sphere in \( R^{n-1} \times 0 \). Therefore, there is a homeomorphism \( g \) of \( R^{n-1} \times 0 \) onto itself such that \( gh(\hat{D}^{n-1}) \) is the standard unit \( n-2 \) sphere \( S_1^{n-2} \) in \( R^{n-1} \times 0 \). \( g \) extends naturally to a homeomorphism \( G \) of \( R^{n-1} \times [0, 1] \) onto itself by \( G(x, t) = (g(x), t) \). Then \( k = Gh \) is a homeomorphism of \( N^n \) onto \( R^{n-1} \times [0, 1] \) such that \( k(D^{n-1}) \) is the standard unit \( n-2 \) sphere \( S_1^{n-2} \) in \( R^{n-1} \times 0 \). \( g \) extends uniquely to a homeomorphism \( j \) of \( R^{n-1} \times [0, 1] \) onto \( L^n \) such that \( j(R^{n-1} \times 1) = (S^{n-1} \times 1) - q' \) and \( j(R^{n-1} \times 0) = [(S^{n-1} \times 0) - \text{Int}(B^{n-1})] \cup (C' - q') \) and \( j(S_1^{n-2}) = \hat{B}^{n-1} \). Then \( f = jk \) is a homeomorphism of \( N^n \) onto \( L^n \) such that \( f(\hat{D}^{n-1}) = \hat{B}^{n-1} \). \( f \) extends uniquely to a homeomor-
phism $F$ of $N^n \cup p$ onto $L^n \cup q'$ by taking $F(p) = q'$.

Finally, let $M$ be the submanifold of $S^n$ bounded by $S_1^{n-1} \cup S_2^{n-1}$. Since $F(C) = C'$, $F$ extends to a homeomorphism of $M$ onto $A^n$ by extending first to take $D^{n-1}$ onto $B^{n-1}$ and finally extending to take the $n$-cell bounded by $C \cup D^{n-1}$ onto $F^n$. Thus, $M$ is homeomorphic to $S^{n-1} \times [0, 1]$ and the theorem is proved.

It does not seem obvious that the Annulus Conjecture implies the Slab Conjecture for $n > 3$.

References


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